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THEORY OF UNDERWATER EXPLOSION BUBBLES

By

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Prepared under
Navy Contract N6ori-201 Task Order No. 1
By the
Institute for Mathematics and Mechanics
New York University

September, 1947

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PREFACE

The present report by Dr. Bernard Friedman was prepared under Navy Contract N6ori-201, Task Order No. 1. It supplements and greatly extends previous work on underwater explosions, in particular AMP Report 37.1R, Studies on the Gas Bubble Resulting from Underwater Explosions: On the Best Location of a Mine Near the Sea Bed, written by Dr. Bernard Friedman and Dr. Max Shiffman.

In preparing the present paper Dr. Friedman was supported by junior members of the staff, particularly by Miss Marie Johnson.

R. Courant

THEORY OF UNDERWATER EXPLOSION BUBBLES

Introduction

The theory of the behavior of a gas bubble produced by an underwater explosion has reached such a state of completeness that it seems desirable to give a unified account of some of the main results. When an explosive is detonated under water, a shock wave is first emitted and then the gaseous products of the explosion expand under the influence of their high internal pressure. Because of its inertia, the gas bubble overexpands to a very low pressure and then the hydrostatic pressure of the water recompresses the bubble to nearly its original size. At this stage the bubble starts expanding again and a pressure pulse is emitted. This process of expansion and contraction may occur five or six times before the bubble breaks up and dissolves or escapes from the water.

The behavior of the bubble is affected by the presence of surfaces, such as the bottom of the water, the walls of a tank, the target or the air-water surface. Normally, the bubble would tend to move upward since it is lighter than the surrounding water. However, the free surface of the water repels the bubble while a rigid surface of any kind such as a wall or a bottom attracts it. In some cases these effects may be so strong that the bubble actually moves downward.

Besides influencing the motion of the bubble, the surfaces also change the period of oscillation and modify the size of the peak pressures produced. The theory

presented here explains these effects and gives a quantitative estimate of their size. The values predicted for the period show excellent agreement with those obtained experimentally, while the values predicted for the peak pressure and the distance moved by the bubble show only fair agreement.

An important application of the theory is the determination of the amount of energy left in the bubble after the shock wave has passed. Also the energy in successive bubble oscillations can be found. There is a large amount of energy dissipated in the transition from shock wave stage to bubble motion and also between successive bubble oscillations, which cannot be explained on the basis of the energy radiated by the pressure pulse. The explanation of this large energy dissipation is still unknown.

Outline

As developed in this paper the theory is an extension of that given in AMP Report 37.1R, Studies on the Gas Bubble Resulting from Underwater Explosions: On the Best Location of a Mine Near the Sea Bed. There, the motion of the bubble in the presence of a rigid bottom was investigated and it was shown that the exact theory could be successfully approximated by the addition of a term to the kinetic energy. In this paper we show that the effects of surface, bottom, walls, targets, etc., can all be approximated in the same way by the addition of a suitable term to the kinetic energy. The evaluation of this term depends upon the solution of an "electrostatic problem." In Section IV we work out in detail the case of a bubble between a free surface and a bottom. Other cases can be treated in the same way.

Section I presents a collection of formulas and a summary of methods which can be used to predict the period of oscillation of the bubble, the distance its

center moves during the first oscillation, the maximum and minimum radius of the bubble and, finally, the peak pressure emitted by the bubble. The formulas are given in terms of certain integrals which can be evaluated by the method given in Report 37.1R. Numerical integration of the differential equations of motion of the bubble is completely unnecessary. For convenience Figures 1-6 contain graphs of these integrals for the most frequently occurring values of the parameters.

Section III contains a discussion of these formulas and a short indication of their proof. A careful analysis is made of the dependence of the parameters in the bubble motion upon the properties of the explosive. It is shown that, by a study of the periods of bubbles placed at different depths, it is possible to determine the amount of energy left in the bubble after the shock wave has passed and also to determine the exponent in the equation for the adiabatic expansion of the explosive. This seems to be one of the very few methods by which this exponent can be found.

A similar procedure can be used to determine the amount of energy left in the second bubble oscillation. The experiments indicate that only about 16 percent of the original energy of the explosive remains. Calculations show that the energy radiated by the pressure pulse emitted by the bubble at minimum size is not large enough to explain the energy loss. The explanation of this high dissipation of energy is one of the major unsolved problems of the theory.

As was mentioned before, Section IV contains a solution of the "electrostatic problem" equivalent to the problem of a bubble placed between a rigid bottom and a free surface.

In Section II the theory is applied to the analysis of some experimental data obtained at Woods Hole by Arons

and his co-workers. Three hundred grams of Tetryl were detonated at varying depths below the surface in water 23.5 feet deep. The period of oscillation, the peak pressure and the distance the bubble moved were measured. The agreement between theory and experiment as regards periods is excellent; as regards pressure and distance the bubble moves, the agreement is only fair.

Since the agreement as regards periods is one of the outstanding successes of the theory, it seems worthwhile elaborating on it. Let $\bar{T}$ be the period of oscillation of the bubble at a depth H feet beneath the surface. In Section II it is shown that $\bar{T}(H + 33)^{5/6}$ varies linearly with $\varphi_1(H + 33)^{-1/3}$ where φ_1 is a complicated function of H, so that we have

$$\bar{T}(H + 33)^{5/6} = \alpha + \beta\varphi_1(H + 33)^{-1/3}$$

where α and β are constants independent of H. It is also shown that $\alpha = C_1E^{1/3}$ and $\beta = C_2E^{2/3}$ where E is the amount of energy left in the bubble after the shock wave has passed and C_1 and C_2 are two constants depending upon the exponent in the equation for the adiabatic expansion of the explosive.

From the experimental data, $\bar{T}(H + 33)^{5/6}$ and $\varphi_1(H + 33)^{-1/3}$ are calculated and then the constants α and β are determined by the method of least squares. From the values of α and β the amount of energy left in the bubble and the adiabatic constant can be determined. In the particular case considered in Section II it is found that about 48 percent of the original energy of the explosive is left in the bubble.

Assumptions

As in Report 37.1R we shall idealize the problem by making the following assumptions:

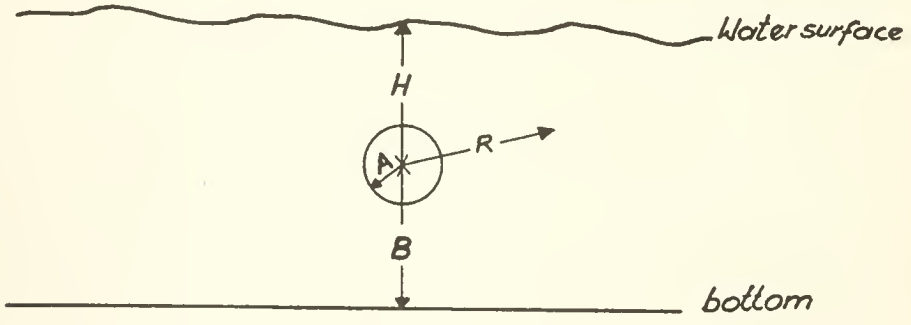
1. The water is an ideal incompressible fluid.
2. The bubble remains spherical in shape.
3. The gas inside the bubble expands adiabatically.

Section I

Summary of Formulas

Units of Length and Time

Let A be the radius of the bubble at time T, B the distance of its center from the bottom, and H the distance from the surface. We put $Z = H + Z^*$, where Z^* is the height of the water column whose pressure equals atmospheric pressure, so that the hydrostatic pressure at



the center of the bubble is $\rho g Z$.

Let E be the amount of energy left in the bubble after the shock wave has passed and let M be the mass of the explosive. We write $E = rQM$ so that rQ is the amount of energy per unit mass of explosive left in the bubble after the shock wave has passed.

All the formulas will be given in dimensionless terms. The unit of length is

$$L = (3E/4\pi\rho g Z_o)^{1/3} \tag{1.1}$$

where Z_o is the value of Z at the time of detonation. The unit of time is

(1.2) $$C = L(2gZ_o/3)^{-1/2}.$$

Note that the unit of velocity is proportional to the square root of Z_o, for

(1.3) $$L/C = (2gZ_o/3)^{1/2}.$$

Parameters in the Bubble Motion

If the pressure and volume of a unit mass of the gas formed by the explosive are connected by the adiabatic relation:

(1.4) $$PV^{\gamma} = K,$$

then the internal energy of the bubble when expressed in non-dimensional terms will depend upon a parameter k which is defined as follows:

(1.5) $$k = K(E/M)^{-\gamma}(\rho g Z_o)^{\gamma-1}/(\gamma - 1).$$

Since $\gamma - 1$ is very close to zero, the value of k will hardly change when Z_o varies slightly. In most cases, therefore, we shall be able to treat k as constant independent of Z_o.

The effects of free surface, gravity and bottom are expressed with the help of a function φ_1. We put

(1.6) $$\begin{aligned} \varphi_1 &= -[f(x) + \log 2]/(H + B) \\ &= -\mathcal{F}(x)/H \end{aligned}$$

where

(1.7) $$x = \frac{B - H}{B + H}$$

and

(1.8) $$f(x) = 2x \sum_{o}^{\infty} \frac{(-)^n}{(2n + 1)^2 - x^2}$$

and

$$(1.9)\qquad \mathcal{F}(x) = \frac{x}{1+x} + (1-x)\left[\sum_{1}^{\infty} \frac{x(-)^n}{(2n+1)^2 - x^2} + \frac{\log 2}{2}\right].$$

Tables and graphs of the functions $f(x)$ and $\mathcal{F}(x)$ are given in the appendix (see Figures 13, 14 and 15 and Tables 3 and 4). Note that if the water is infinitely deep, $B = \infty$ so that from (1.7), (1.9) and (1.6) we have

$$(1.10)\qquad x = 1, \quad \mathcal{F}(x) = 1/2, \quad \varphi_1 = -1/2H.$$

Bubble Quantities at Time of Maximum Size

Small letters, a, b, h, etc., will be used to represent the non-dimensional values of the quantities represented by the corresponding capital letters. The subscript zero attached to these quantities will indicate their value at the beginning of the bubble motion, the subscript one their value at the time of maximum size, and a bar attached to them will indicate their value at the time of recompression when the bubble size is a minimum.

The non-dimensional time from the beginning of the bubble motion to the stage of maximum size is

$$(1.11)\qquad t_1 = I_1 + \frac{1}{2} I_2 L\varphi_1,$$

where

$$(1.12)\qquad I_1 = \int_{a_o}^{a_1} \frac{a^{3/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}}$$

and

$$(1.13)\qquad I_2 = \int_{a_o}^{a_1} \frac{a^{5/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}}\,.$$

Here, a_o and a_1 denote the smallest and largest roots, respectively, of the denominator. Graphs of I_1 and I_2 for the most frequently occurring values of k and γ are given in Figures 1 and 2.

The vertical momentum at the time of maximum size is

$$(1.14) \quad s_1 = L \frac{\partial \varphi_1}{\partial B}(I_3 - \tfrac{1}{2} L\varphi_1 I_4) + z_o^{-1}(I_5 + \tfrac{1}{2} L\varphi_1 I_6)$$

where

$$(1.15) \quad I_3 = \int_{a_o}^{a_1} a^{5/2}\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}\, da,$$

$$(1.16) \quad I_4 = \int_{a_o}^{a_1} a^{7/2}\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}\, da,$$

$$(1.17) \quad I_5 = \int_{a_o}^{a_1} \frac{a^{9/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}},$$

$$(1.18) \quad I_6 = \int_{a_o}^{a_1} \frac{a^{11/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}}.$$

The maximum radius of the bubble, a_1, is the root near 1 of the equation:

$$(1.19) \quad a^3 + ka^{-3(\gamma-1)} + 3s_1^2/2a^3 = 1.$$

A good approximation to a_1 is given by the formula:

$$(1.20) \quad a_1 = 1 - \frac{k}{3} - (\frac{\gamma - 1}{3} + \frac{1}{9})k^2 - \frac{3}{2} s_1^2.$$

Bubble Quantities at Time of Minimum Size

The time, $\bar{t}$, from the beginning of the bubble motion to the stage of minimum size is twice the time to the stage of maximum size, that is,

$$(1.21) \quad \bar{t} = 2t_1.$$

The vertical momentum, $\bar{s}$, at the time of minimum size is twice the vertical momentum at the time of maximum size, that is,

$$\bar{s} = 2s_1. \tag{1.22}$$

The minimum radius of the bubble, $\bar{a}$, is the root near zero of the equation:

$$a^3 + ka^{-3(\gamma-1)} + 3\bar{s}^2/2a^3 = 1. \tag{1.23}$$

If we put

$$\xi = k^{1/(\gamma-1)}, \quad \varepsilon = 3\bar{s}^2/2\xi \tag{1.24}$$

and

$$\bar{a}^3 = \xi u, \tag{1.25}$$

then u is the root of the equation:

$$u = u^{2-\gamma} + \varepsilon. \tag{1.26}$$

A graph of u for various values of ε and γ is given in Figure 7.

The non-dimensional distance the bubble moves downward is given approximately by the formula:

$$\Delta b = 3\bar{s}(I_7 + \tfrac{1}{2} L\varphi_1 I_8) \tag{1.27}$$

where

$$I_7 = \int_{\bar{a}}^{a_1} \frac{a^{-3/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3\bar{s}^2/2a^3}} \tag{1.28}$$

and

$$I_8 = \int_{\bar{a}}^{a_1} \frac{a^{-1/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3\bar{s}^2/2a^3}}. \tag{1.29}$$

The non-dimensional peak pressure at the time of minimum size is

$$(1.30)\quad (a^2\dot{a})^{\cdot} = \xi^{-2/3}\, u^{-5/3}[\varepsilon^2 + \frac{3(\gamma-1)}{2}\, u^{2-\gamma}][1 + \bar{a}L\varphi_1]^{-1}$$

where

$$(1.31)\qquad u = \bar{a}^{-3}\, \xi^{-1}.$$

Actual Values of Bubble Quantities

If $\bar{T}$ is the actual period from the beginning of the bubble motion to the time of minimum size, then

$$\bar{T} = c\bar{t}$$

or using (1.21), (1.11), (1.1) and (1.2) we find that

$$(1.32)\qquad \bar{T} = \alpha Z_o^{-5/6} + \beta\varphi_1 Z_o^{-7/6}$$

where

$$(1.33)\qquad \alpha = 2I_1(2g/3)^{-5/6}\, (2\pi\rho)^{-1/3}\, E^{1/3},$$

$$(1.34)\qquad \beta = I_2(2g/3)^{-7/6}\, (2\pi\rho)^{-2/3}\, E^{2/3}.$$

Note that

$$(1.35)\qquad \bar{T}Z_o^{5/6} = \alpha + \beta\varphi_1 Z_o^{-1/3},$$

so that $\bar{T}Z_o^{5/6}$ is a linear function of $\varphi_1 Z_o^{-1/3}$. In a later section we shall use this formula to find α and β and then the value of E.

The maximum and minimum radii are found from formulas (1.20) and (1.23) after multiplication by the unit of length, L, given in (1.1).

The peak pressure of the bubble at a distance R from the center of the bubble is

$$P = \frac{2}{3}\rho g Z_o \cdot \frac{L}{R}(a^2\dot{a})^{\cdot} \tag{1.36}$$

where the value of $(a^2\dot{a})^{\cdot}$ is given by formula (1.30).

At the time of maximum size the pressure in the bubble is

$$P_1 = \frac{2}{3}\rho g Z_o \frac{3(1 - 5a_1^3)}{8a_1^2} \tag{1.37}$$

Section II

Analysis of Experimental Data

In this section some experimental data obtained by A. Arons and his co-workers at Woods Hole Oceanographic Institution will be analyzed. It will be shown that the theoretical values of the period agree very well with the experimental data while the values for pressure and migration do not agree as well.

The experiments consisted of exploding three hundred grams Tetryl at various depths beneath the surface in water 23.5 feet deep. Table 1 shows the values observed for the period of bubble oscillation.

Before these data can be compared with the theoretical results, the value of E must be known. We shall find E by fitting a straight line to the points obtained when $\bar{T}Z_o^{5/6}$ is plotted against $\varphi_1 Z_o^{-1/3}$. Table 2 gives the values of $\bar{T}Z_o^{5/6}$ and the corresponding values of $\varphi_1 Z_o^{-1/3}$.

A straight line is fitted to these values by the method of least squares. The equation obtained is

$$\bar{T}Z_o^{5/6} = 3.876 - 16.96\, \varphi_1 Z_o^{-1/3}. \tag{2.1}$$

Figure 8 shows how closely the straight line fits the data. This closeness of fit is a confirmation of the theory.

When equation (2.1) is compared with (1.35), we find that

$$\alpha = 3.876, \quad \beta = 16.96. \tag{2.2}$$

Table 1

H (feet)	T (seconds)	H (feet)	T (seconds)
3.5	.1576, .1577, .1582	16.4	.1509
4.0	.1590, .1598, .1602, .1610	17.3	.1516
4.5	.1605, .1608, .1613	18.5	.1475
5.0	.1619, .1621, .1624	18.8	.1480
5.5	.1632, .1637, .1641	19.0	.1455, .1480
6.0	.1625, .1628, .1630	19.1	.1496
6.5	.1643, .1648, .1650	19.2	.1474
7.0	.1629, .1643, .1646, .1651	19.6	.1444
8.0	.1619, .1638, .1640, .1646	19.7	.1456, .1468, .1531
10.0	.1602, .1604, .1609, .1614, .1615	19.8	.1437, .1450, .1456
12.0	.1560, .1564, .1566, .1570, .1591	20.2	.1491
14.0	.1531, .1534, .1543	20.4	.1521
15.4	.1511		

Table 2

H	$TZ_o^{5/6}$	$\varphi_1 Z_o^{-1/3}$	H	$TZ_o^{5/6}$	$\varphi_1 Z_o^{-1/3}$	H	$TZ_o^{5/6}$	$\varphi_1 Z_o^{-1/3}$
3.5	3.158	.0428	7.0	3.524	.0198	15.4	3.830	.000620
	3.160			3.554				
	3.170			3.560		16.4	3.893	-.00210
				3.571		17.3	3.969	-.00525
4.0	3.223	.0372						
	3.239		8.0	3.575	.0166	18.5	3.938	-.0108
	3.247			3.617				
	3.263			3.621		18.8	3.971	-.0122
				3.634				
4.5	3.290	.0332				19.0	3.915	-.0140
	3.296		10.0	3.680	.0121		3.933	
	3.307			3.684				
				3.696		19.1	4.032	-.0148
5.0	3.353	.0292		3.707				
	3.357			3.710		19.2	3.980	-.0157
	3.363							
						19.6	3.923	-.0188
			12.0	3.722	.00710			
5.5	3.419	.0262		3.732		19.7	3.963	-.0197
	3.430			3.736			3.996	
	3.438			3.746			4.167	
				3.796				
6.0	3.442	.0238				19.8	3.917	-.0204
	3.448		14.0	3.788	.00385		3.953	
	3.452			3.795			3.969	
				3.817				
6.5	3.516	.0216						
	3.527					20.2	4.091	-.0264
	3.531					20.4	4.174	-.0288

Formulas (1.33) and (1.34) show that α and β depend on the value of E and of I_1 and I_2. If we consider the ratio α^2/β we have

$$(2.3) \quad \frac{\alpha^2}{\beta} = 4I_1^2 I_2^{-1}(2g/3)^{-1/2} \quad \text{or} \quad \frac{I_1^2}{I_2} = 1.155\,\frac{\alpha^2}{\beta}$$

which is independent of E. A graph of I_1^2/I_2 for different values of k and γ is given in Figure 9.

Three physical quantities are needed to fully describe the bubble motion--the values of K, γ and E. Since we have only two constants α and β determined, we must assume the value of one of the three quantities. We shall assume that $\gamma = 1.25$, the value proposed by Jones for T.N.T., and then by the use of Figure 9 we find that $k = .23$. From this value of k, using Figures 1 and 2 and formulas (1.33) and (1.34), we find that $rQ = 500$ calories/gram. Since the detonation energy of Tetryl is 1060 calories/gram, this value for rQ is another confirmation of the theory.

Since we know the value of $E = rQM$, we can use (1.1) and get

$$(2.4) \qquad L = 13.88\ (W/Z_o)^{1/3}$$

where L is in feet, W in pounds and Z_o in feet, and

$$C = 2.997\ W^{1/3} Z_o^{-5/6}.$$

Since k varies with depth as $Z_o^{\gamma-1}$, the value, $k = .23$, is really an average value over the range $Z_o = 33$ to $Z_o = 56.5$. If we assume that this value of k corresponds to a depth halfway between the bottom and the surface of the water, we find that k varies from .213 to .244. This change in k can be neglected in the calculation of the period, momentum and migration. However, in the calculation of the peak pressure it must be taken into account since the pressure varies as $k^{-2/3(\gamma-1)}$.

Figure 10 shows how the periods calculated from the equation (2.1) compare with the observed results. The agreement is excellent.

Using k = .23 and formulas (1.27) and (1.28) the migration was found by numerical integration of I_7 for each value Z_o. The calculated results are compared with the observed values in Figure 11. The agreement seems to be fair.

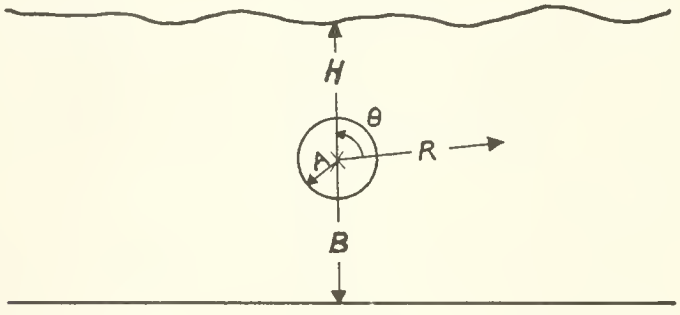
The peak pressure is calculated from formulas (1.36) and (1.30). If we use the values of k and γ determined by the period measurements, the calculated pressures are about one-half the observed pressure. Theoretically, it would be possible to use one experimentally determined value of pressure with the two constants α and β to solve for the three quantities rQ, k and γ. This was done but it did not lead to consistent results. It seems likely that the value of γ at the time of minimum size may be different from the value of γ during the time of expansion and contraction.

Despite this difficulty with the magnitude of the pressure, the variation of pressure with depth is given approximately by formula (1.30). If we multiply the calculated pressures by a constant chosen so that the calculated pressure agrees with that experimentally observed at the depth of 18.5 feet, we get Figure 12.

Section III

Derivation of Formulas

Consider the motion of a gas bubble in water of finite depth, taking into account the effects of the bottom, the free surface and gravity. As before, we let A be the radius of the bubble at the time T, B the distance of its center from the bottom, and H the distance from the surface. We put $Z = H + Z^*$ where Z^* is the height of the



water column whose pressure equals atmospheric pressure, so that the pressure at the center of the bubble is $\rho g Z$.

Just as in Report 37.1R we find that if we use coordinates R and θ to describe the motion of the water, the velocity potential describing the flow is

$$\Phi = \frac{A^2 A'}{R} + \frac{A^3 B'}{2} \frac{\cos\theta}{R^2} + \Phi_1 \tag{3.1}$$

where Φ_1 is the "image" potential necessary to satisfy the boundary conditions on the bottom and on the surface. The primes indicate time derivatives.

By classical hydrodynamics we can show (see Report 37.1R) that the kinetic energy of the water is given by

$$(3.2)\quad 2\pi\rho A^3[A'^2(1 + A\varphi_1) - 2A^2\varphi_2 A'B' + \frac{1}{6}B'^2(1 + A^3\varphi_3)]$$

where φ_1, φ_2 and φ_3 are functions of A, B and H which represent the effect of the surface and the bottom. Note that φ_1, φ_2, φ_3 are of the dimensions $(\text{length})^{-1}$, $(\text{length})^{-2}$, $(\text{length})^{-3}$ respectively.

The potential energy of the displaced water is equal to the volume of the bubble multiplied by the hydrostatic pressure at the center of the bubble, that is, to $4\pi A^3\rho gZ/3$.

Assume the pressure and volume of one unit mass of the gas in the bubble under an adiabatic change are connected by the relation

$$(3.3)\qquad PV^\gamma = K.$$

If M is the mass of the explosive, then for the actual bubble

$$PV^\gamma = KM^\gamma$$

and the internal energy of the gas when the bubble has radius A is

$$(3.4)\qquad \begin{aligned} G(A) &= \int PdV = KM^\gamma V^{1-\gamma}/(\gamma - 1) \\ &= KM^\gamma(\tfrac{4}{3}\pi A^3)^{1-\gamma}/(\gamma - 1). \end{aligned}$$

By our assumption the total energy, E, left in the bubble after the shock wave has passed is equal to the sum of the kinetic energy of the water and the potential energy of the displaced water and the internal energy of the gas. We have, then

$$(3.5)\qquad \begin{aligned} &2\pi\rho A^3[A'^2(1 + A\varphi_1) - 2A^2\varphi_2 A'B' + \tfrac{1}{6}B'^2(1 + A^3\varphi_3)] \\ &\quad + \tfrac{4}{3}\pi A^3\rho gZ + G(A) = E. \end{aligned}$$

If we use L and C defined in (1.1) and (1.2) as units, equation (3.5) can be written in terms of non-dimensional quantities as follows:

$$a^3[\dot{a}^2(1 + aL\varphi_1) - 2\dot{a}\dot{b}a^2L^2\varphi_2 + \frac{1}{6}\dot{b}^2(1 + a^3L^3\varphi_3)] + a^3zz_o^{-1} + ka^{-3(\gamma-1)} = 1, \tag{3.6}$$

where $ka^{-3(\gamma-1)}$ is the internal energy of the gas expressed in non-dimensional quantities. k is defined in equation (1.5).

It was shown in Report 37.1R that the period can be found by assuming b is constant so that $\dot{b} = 0$. Equation (3.6) reduces to

$$a^3\dot{a}^2(1 + aL\varphi_1) + a^3 + ka^{-3(\gamma-1)} = 1. \tag{3.7}$$

If we approximate $(1 + aL\varphi_1)^{1/2}$ by $1 + aL\varphi_1/2$, equation (3.7) can be solved for $\dot{a}$ and we get

$$\frac{a^{3/2}(1 + aL\varphi_1/2)da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}} = dt. \tag{3.8}$$

Let

$$I_1 = \int_{a_o}^{a_1} \frac{a^{3/2}\,da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}}, \tag{3.9}$$

$$I_2 = \int_{a_o}^{a_1} \frac{a^{5/2}\,da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}}, \tag{3.10}$$

where a_o and a_1 are respectively the smallest and largest zeros of the denominator. We find then that the time from beginning to maximum is

$$t_1 = I_1 + \frac{1}{2} I_2L\varphi_1, \tag{3.11}$$

which is formula (1.11). The total period from beginning to minimum will be twice t_1.

Momentum of the Bubble

The vertical momentum, s, of the bubble is equal to $a^3\dot{b}/3$. Using the Lagrangian equation associated with the b coordinate, we find that

$$\frac{d}{dt}(a^3\dot{b}/3) = a^4\dot{a}^2 L \frac{\partial \varphi_1}{\partial b} + a^3 z_o^{-1}. \tag{3.12}$$

Integration gives the value of s at the maximum as

$$s_1 t = 2\left[\int_{t_o}^{t_1} a^4\dot{a}^2 L \frac{\partial \varphi_1}{\partial b}\, dt + z_o^{-1} \int_{t_o}^{t_1} a^3 dt\right].$$

Using equations (3.7) and (3.8) and the fact that

$$\dot{a}dt = da$$

we express s_1 as integrals over a. We have

$$s_1 = L \frac{\partial \varphi_1}{\partial b} \int_{a_o}^{a_1} a^{5/2}(1+aL\varphi_1)^{-1/2}\sqrt{1-a^3-ka^{-3(\gamma-1)}}\, da$$

$$+ z_o^{-1} \int_{a_o}^{a_1} \frac{a^{9/2}(1 + aL\varphi_1/2)da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}} \tag{3.13}$$

$$= L \frac{\partial \varphi_1}{\partial b}(I_3 - \frac{1}{2} L\varphi_1 I_4) + z_o^{-1}(I_5 + \frac{1}{2} L\varphi_1 I_6)$$

where

$$I_3 = \int_{a_o}^{a_1} a^{5/2}\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}\, da,$$

$$I_4 = \int_{a_o}^{a_1} a^{7/2}\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}\, da,$$

(3.14)

$$I_5 = \int_{a_o}^{a_1} \frac{a^{9/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}},$$

$$I_6 = \int_{a_o}^{a_1} \frac{a^{11/2}\, da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)}}}.$$

These equations are the same as (1.14)-(1.18).

The momentum can be used to give us an approximation to the energy equation (3.6) which takes into account the fact that b does change. We shall neglect the terms $a^2L^2\varphi_2$ and $a^3L^3\varphi_3$ which are of second and third order, respectively, as compared to the first order term $aL\varphi_1$. Now, since

$$\dot{b} = 3s/a^3,$$

equation (3.6) becomes

$$a^3\dot{a}^2(1 + aL\varphi_1) + a^3 + ka^{-3(\gamma-1)} + 3s^2/2a^3 = 1. \tag{3.15}$$

If s were known as a function of a, this could be integrated:

$$\int \frac{a^{3/2}(1 + aL\varphi_1/2)da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3s^2/2a^3}} = t. \tag{3.16}$$

The vertical distance the bubble moves can be found from this formula since

$$\dot{b} = 3s/a^3$$

so that

$$(3.17)\quad \Delta b = 3\int sdt/a^3 = 3\int \frac{sa^{-3/2}(1 + aL\varphi_1/2)da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3s^2/2a^3}}\,.$$

Since s is given as the indefinite integral of the right side of (3.12), Δb is really a double integral but we can change it to a single integral by the following argument:

During the first part of the bubble motion, that is, until the bubble reaches its maximum size, s increases from zero to s_1. During the second part of the motion, from the bubble maximum to the bubble minimum, s increases from s_1 to $\bar{s} = 2s_1$. The momentum at any time during the second half of the motion is $\bar{s}$ minus the momentum developed as the bubble contracts from its given size to its minimum size. Because of the symmetry of the bubble motion, this minus momentum is the same as the momentum developed when the bubble increases from its initial size to the given size. We may then conclude that the distance moved during the bubble expansion just balances the distance which would be produced by the minus momentum so that the total distance moved is given by one term

$$(3.18)\quad \begin{aligned}\Delta b &= 3\bar{s}\int_{a_1}^{\bar{a}} \frac{a^{-3/2}(1 + aL\varphi_1/2)da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3\bar{s}^2/2a^3}} \\ &= 3\bar{s}(I_7 + \tfrac{1}{2} L\varphi_1 I_8),\end{aligned}$$

$$(3.19)\quad I_7 = \int_{a_1}^{\bar{a}} \frac{a^{-3/2}\,da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3\bar{s}^2/2a^3}}\,,$$

$$(3.20)\quad I_8 = \int_{a_1}^{\bar{a}} \frac{a^{-1/2}\,da}{\sqrt{1 - a^3 - ka^{-3(\gamma-1)} - 3\bar{s}^2/2a^3}}\,.$$

This proves (1.27).

Radius of the Bubble

The maximum radius of the bubble is found by putting $\dot{a} = 0$ in equation (3.15) and solving

$$a^3 + ka^{-3(\gamma-1)} + 3s_1^2/2a^3 = 1 \tag{3.21}$$

for the root near one. If we expand a in powers of k and solve (3.21) we find that

$$a_1 = 1 - \frac{k}{3} - (\frac{\gamma - 1}{3} + \frac{1}{9})k^2 - \frac{3s_1^2}{2} . \tag{3.22}$$

This is the same as formula (1.20).

To find the minimum radius of the bubble we put $\dot{a} = 0$ in equation (3.15) and consider it at the time of minimum size. The equation reduces to

$$a^3 + ka^{-3(\gamma-1)} + 3\bar{s}^2/2a^3 = 1 \tag{3.23}$$

and the root near zero is the minimum radius. Note that (3.23) is the same as (3.21) except for the value of the momentum. The equation simplifies if we put

$$\xi = k^{1/(\gamma-1)}, \quad \bar{a}^3 = \xi u, \quad \varepsilon = 3\bar{s}^2/2\xi \tag{3.24}$$

and neglect that a^3 term, since a is near zero. We have

$$u^{-(\gamma-1)} + \varepsilon u^{-1} = 1$$

or

$$u = u^{2-\gamma} + \varepsilon . \tag{3.25}$$

Peak Pressure

The pressure in the water is given by Bernoulli's Law as

$$\frac{P}{\rho} = \frac{\partial \Phi}{\partial t} - \frac{1}{2}(\nabla \Phi)^2 .$$

The second term can be neglected compared to the first. Using (3.1) we find that

$$\frac{P}{\rho} = \frac{(A^2A')'}{R} + \frac{(A^3B')'\cos\theta}{2R^2} + \cdots .$$

For moderate distances from the bubble only the first term is important, so that

$$\frac{P}{\rho} = \frac{L^3}{c^2}\frac{(a^2\dot{a})^{\cdot}}{R} . \tag{3.26}$$

The value of $(a^2\dot{a})^{\cdot}$ can be found by multiplying (3.15) by a and then differentiating. This gives

$$2a^2\dot{a}(a^2\dot{a})^{\cdot}(1 + aL\varphi_1) + a^4\dot{a}^2\dot{a}L\varphi_1 + 4a^3\dot{a}$$
$$+ k(4 - 3\gamma)a^{3-3\gamma}\dot{a} - 3s^2a^{-3}\dot{a} = \dot{a}.$$

Since the peak pressure will be found at the time of minimum size, we may put $\dot{a} = 0$ and solve for $(a^2\dot{a})^{\cdot}$. The result is

$$(a^2\dot{a})^{\cdot} = \frac{\frac{1}{2\bar{a}^2} + \frac{3\bar{s}^2}{2\bar{a}^5} - \frac{k(4 - 3\gamma)}{2}\bar{a}^{1-3\gamma}}{1 + \bar{a}L\varphi_1}$$

$$= \frac{\frac{9\bar{s}^2}{4\bar{a}^5} + \frac{3k(\gamma - 1)}{2}\bar{a}^{1-3\gamma}}{1 + \bar{a}L\varphi_1} \tag{3.27}$$

if we make use of equation (3.23). Using the notation of (3.24) equation (3.27) becomes

$$(a^2\dot{a})^{\cdot} = \frac{1}{\bar{a}^2}\left[\frac{3\varepsilon}{2u} + \frac{3(\gamma - 1)}{2}u^{1-\gamma}\right]$$

$$= \frac{3(\gamma - 1)}{2\xi^{2/3}u^{2/3}}\left[\frac{\varepsilon}{(\gamma - 1)u} + u^{1-\gamma}\right]. \tag{3.28}$$

Notice that since u is near one the bracket does not change much as k changes.

We may write formula (3.26) for the pressure in a more convenient form as follows:

$$P = \frac{L^2}{C^2}\frac{L}{R}(a^2\dot{a})^{\cdot} = \frac{2}{3}\rho g Z_o \cdot \frac{L}{R}(a^2\dot{a})^{\cdot}$$

(3.29)

$$= \frac{(\gamma - 1)\rho g Z_o}{\xi^{2/3}\, u^{2/3}}\left[\frac{\varepsilon}{(\gamma - 1)u} + u^{1-\gamma}\right]\frac{L}{R}$$

by formulas (1.3) and (3.28).

Replace L by its value given in (1.1); then

$$P = \frac{(\gamma - 1)\rho g Z_o}{\xi^{2/3}\, u^{2/3}\, R}\left[\frac{\varepsilon}{(\gamma - 1)u} + u^{1-\gamma}\right]\left(\frac{3E}{4\pi\rho g Z_o}\right)^{1/3}. \tag{3.30}$$

This formula can be combined with the formulas (1.33) and (1.34) for α and β to solve for the three parameters of the bubble motion: rQ, K and γ.

Suppose, for example, that we know P when the momentum, s, is zero. This means that $\varepsilon = 0$ and $u = 1$. Consider the ratio P/α. We have

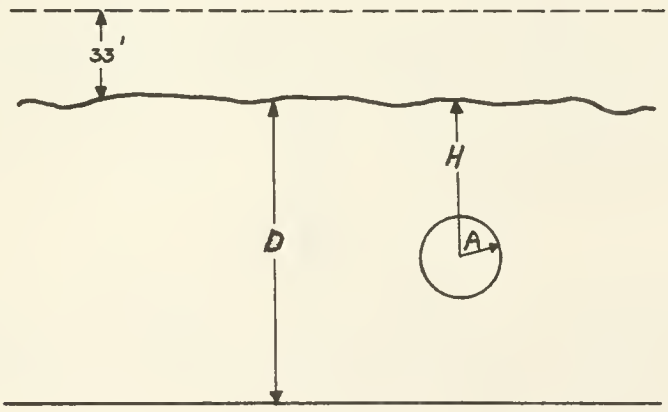
$$\frac{P}{\alpha} = \frac{(\gamma - 1)\rho g\left(\frac{2g}{3}\right)^{1/2} Z_o^{2/3}}{2I_1\, R\, \xi^{2/3}} \tag{3.31}$$

and the right side depends only on k and γ. Combining this with the formula for α^2/β, which also depends only on k and γ, we will be able to solve for these quantities.

Section IV

The Electrostatic Problem

In Section III functions φ_1, φ_2 and φ_3 were introduced into the energy equation to take account of the presence of free and rigid surfaces. However, the results obtained depended only on the function φ_1. This function φ_1 can be evaluated, to a first approximation, by finding the kinetic energy of water motion due to a fixed expanding bubble. Consider the velocity potential for a fixed expanding bubble, assumed spherical, located at a distance H beneath the surface in water whose depth is D feet. If we take spherical coordinates, R, θ, Ψ



with pole at the center of the bubble, the problem can be formulated mathematically as follows:

Find a potential function $\Phi(R,\theta)$ such that

(4.1) $$\frac{\partial \Phi}{\partial R} = \dot{A} \quad \text{on the sphere,}$$

(4.2) $$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on bottom,}$$

(4.3) $$\Phi = 0 \quad \text{on surface.}$$

The last condition is obtained by taking Bernoulli's equation on the surface, assuming that gravity can be neglected, and then neglecting square terms.

Equation (4.1) is satisfied by taking

$$\Phi = \frac{A^2\dot{A}}{R} .$$

This is equivalent to assuming a source of strength $A^2\dot{A}$ at the center of the sphere. We can then satisfy equations (4.2) and (4.3) by reflecting this source in the bottom and in the free surface. It turns out that the image with respect to one boundary must be reflected in the other boundary and this process carried out on the successive images leads to an infinite sequence of images: Let

$$\Phi = A^2\dot{A}\left(\frac{1}{R} + \Phi'\right) \tag{4.4}$$

where Φ' is the potential due to the images. It can be evaluated by a method similar to that in Report 37.1R.

After Φ has been found we get the kinetic energy of the water by evaluating

$$\frac{1}{2}\rho \int \Phi \frac{\partial \Phi}{\partial n}\, dS$$

over all boundaries. Because of conditions (4.2) and (4.3) this reduces to the integral over the sphere.

Using equation (4.1) we have that

$$E = \frac{1}{2}\rho \int_{\text{sphere}} \Phi \frac{\partial \Phi}{\partial n}\, dS = \dot{A} \int \Phi A^2 d\omega = A^2\dot{A} \int \Phi d\omega \tag{4.5}$$

where $d\omega$ is the element of solid angle so that $dS = A^2 d\omega$. By the Mean Value theorem

$$\frac{1}{4\pi} \int \Phi'\, d\omega = \Phi'(0)$$

where $\Phi'(0)$ is the value of the potential Φ' at the center

of the sphere. From equation (4.4) and this result we find that

$$E = 2\pi\rho A^4\dot{A}^2\left(\frac{1}{A} + \Phi'(0)\right)$$

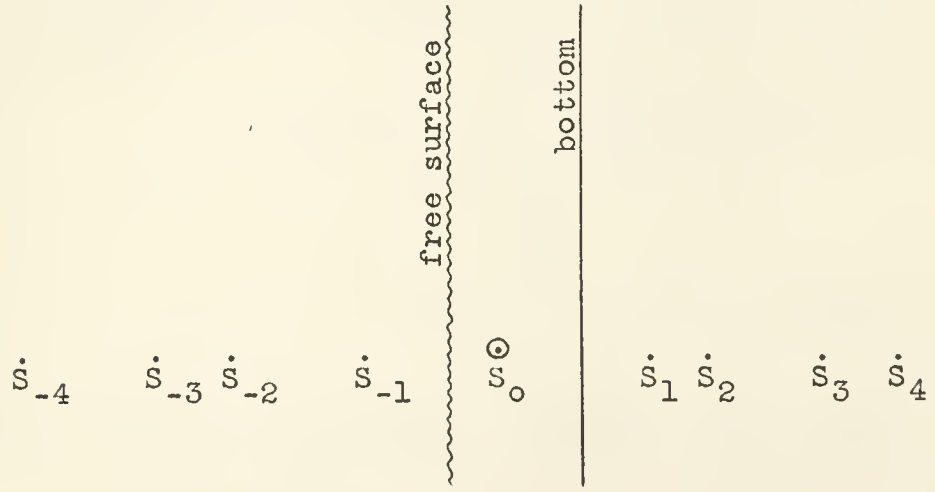
$$= 2\pi\rho A^3\dot{A}^2(1 + A\Phi'(0)) \tag{4.6}$$

so that $\varphi_1 = \Phi'(0)$.

Solution of the Electrostatic Problem

This shows that we need the value of Φ only at the center of the sphere. It will be found by considering the infinite set of images produced by reflecting the original source at the center and its images in the bottom and free surface.

For convenience of representation we shall draw the surfaces vertically. The situation is as follows:



Here, S_0 is the source of strength one at the center of the bubble. S_1, S_{-1}, S_2, S_{-2}, $\cdots$ denote the image sources and sinks. They are obtained as follows:

S_1 is the reflection of S_o in the bottom

S_{-1} is the reflection of S_o in the surface

S_{n+1} is the reflection of S_{-n} in the bottom

S_{-n} is the reflection of S_{n-1} in the surface.

Note that a reflection in the bottom, to satisfy equation (4.2), gives an image of the same strength, while a reflection in the free surface to satisfy equation (4.3) gives an image of the same strength with the opposite sign. (A source of negative strength is to be considered a sink.) We find that

the strength of S_{2n} is $(-)^n$ at a distance $2nD$ from S_o

the strength of S_{2n-1} is $(-)^{n+1}$ at a distance $2nD-2B$ from S_o

the strength of S_{-2n} is $(-)^n$ at a distance $2nD$ from S_o

the strength of S_{-2n-1} is $(-)^{n+1}$ at a distance $2nD+2B$ from S_o.

$\Phi'(0)$, the potential at the center of the bubble due to this collection of sources and sinks can be obtained by combining the effects of S_n and S_{-n}.

The potential at S_o due to S_{2n} and S_{-2n} is

$$\frac{(-)^n}{nD}$$

while that due to S_{2n+1} and S_{-2n-1} is

$$(4.7)\quad (-)^n\left[\frac{1}{(2n+2)D-2B}-\frac{1}{2nD+2B}\right]=\frac{(-)^n\,(4B-2D)}{[(2n+1)D+D-2B][(2n+1)D-(D-2B)]}$$

The total potential due to all the image sources and sinks is, therefore,

$$(4.8)\quad \sum_1^\infty \frac{(-)^n}{nD}+\frac{1}{D}\left(\frac{4B}{D}-2\right)\sum_0^\infty \frac{(-)^n}{(2n+1)^2-\left(1-\frac{2B}{D}\right)^2}$$

or, if we let $\frac{2B}{D} - 1 = x$, this can be written as

$$(4.9) \qquad -\frac{\log 2}{D} + \frac{2x}{D} \sum_{0}^{\infty} \frac{(-)^n}{(2n+1)^2 - x^2} .$$

The series on the right can be expressed in terms of tabulated functions as follows:

$$(4.10) \qquad \begin{aligned} f(x) &= 2 \sum_{0}^{\infty} \frac{(-)^n x}{(2n+1)^2 - x^2} \\ &= 2 \frac{d}{dx}\left[\log \Gamma\left(\frac{1+x}{4}\right) + \Gamma\left(\frac{1-x}{4}\right)\right] - \frac{\pi}{2} \tan \frac{\pi x}{2} \end{aligned}$$

and we have

$$(4.11) \qquad \begin{aligned} \Phi'(0) &= -\frac{1}{D}(f(x) + \log 2) \\ &= -\frac{1}{2B} F(x) = -\frac{1}{2Lb_o} F(x). \end{aligned}$$

Note, that if $D \longrightarrow \infty$, $\Phi'(0) = -\frac{1}{2B}$. By differentiation we find that

$$(4.12) \qquad \frac{\partial \Phi'}{\partial b} = -\frac{1}{D} f'(x) \frac{\partial x}{\partial b} = -\frac{2}{D^2} f'(x).$$

Appendix

Table 3

x	f(x)
0	0
.05	.09184
.10	.1852
.15	.2816
.20	.3829
.25	.4910
.30	.6081
.35	.7379
.40	.8840
.45	1.051
.50	1.246
.55	1.481
.60	1.769
.65	2.136
.70	2.619
.75	3.292
.80	4.298
.85	5.968
.90	9.305
.95	19.31
1.00	∞

Table 4

x	$\mathcal{F}(x)$	x	$\mathcal{F}(x)$
0	.3466		
.05	.3729	-.05	.3157
.10	.3952	-.10	.2794
.15	.4143	-.15	.2366
.20	.4304	-.20	.1862
.25	.4440	-.25	.1264
.30	.4554	-.30	.05528
.35	.4651	-.35	-.03021
.40	.4731	-.40	-.1336
.45	.4796	-.45	-.2593
.50	.4849	-.50	-.4150
.55	.4891	-.55	-.6104
.60	.4924	-.60	-.8608
.65	.4950	-.65	-1.190
.70	.4969	-.70	-1.637
.75	.4982	-.75	-2.274
.80	.4991	-.80	-3.244
.85	.4996	-.85	-4.880
.90	.4999	-.90	-8.181
.95	.49998	-.95	-18.15
1.00	.5	-1.00	$-\infty$

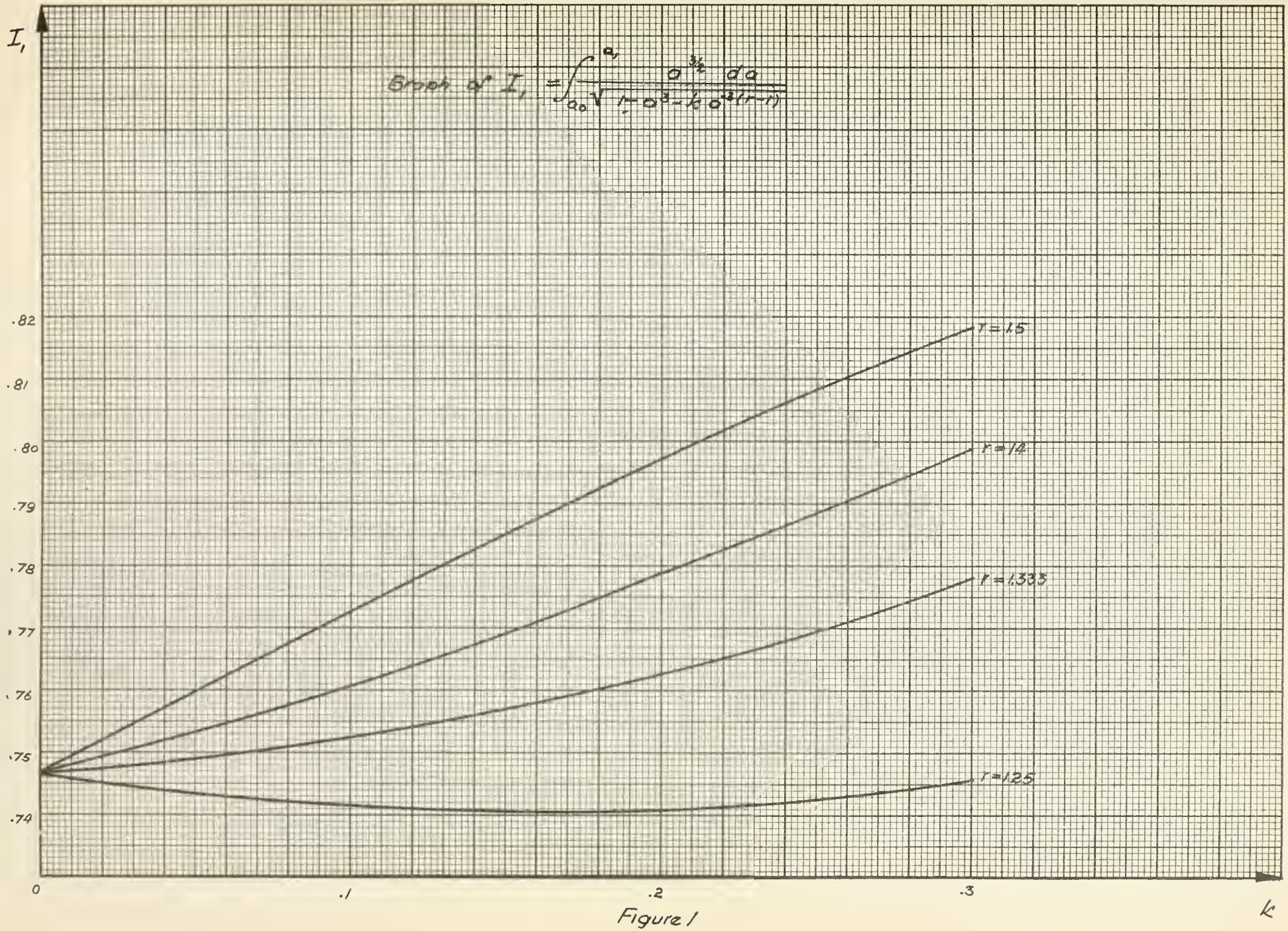

I_1
Graph of $I_1 = \int_{a_0}^{a_1} \frac{a^{3/2}\, da}{\sqrt{1 - a^3 - k\, a^{2(r-1)}}}$
.82
.81
.80
.79
.78
.77
.76
.75
.74
r = 1.5
r = 1.4
r = 1.333
r = 1.25
0
.1
.2
.3
k

Figure 1

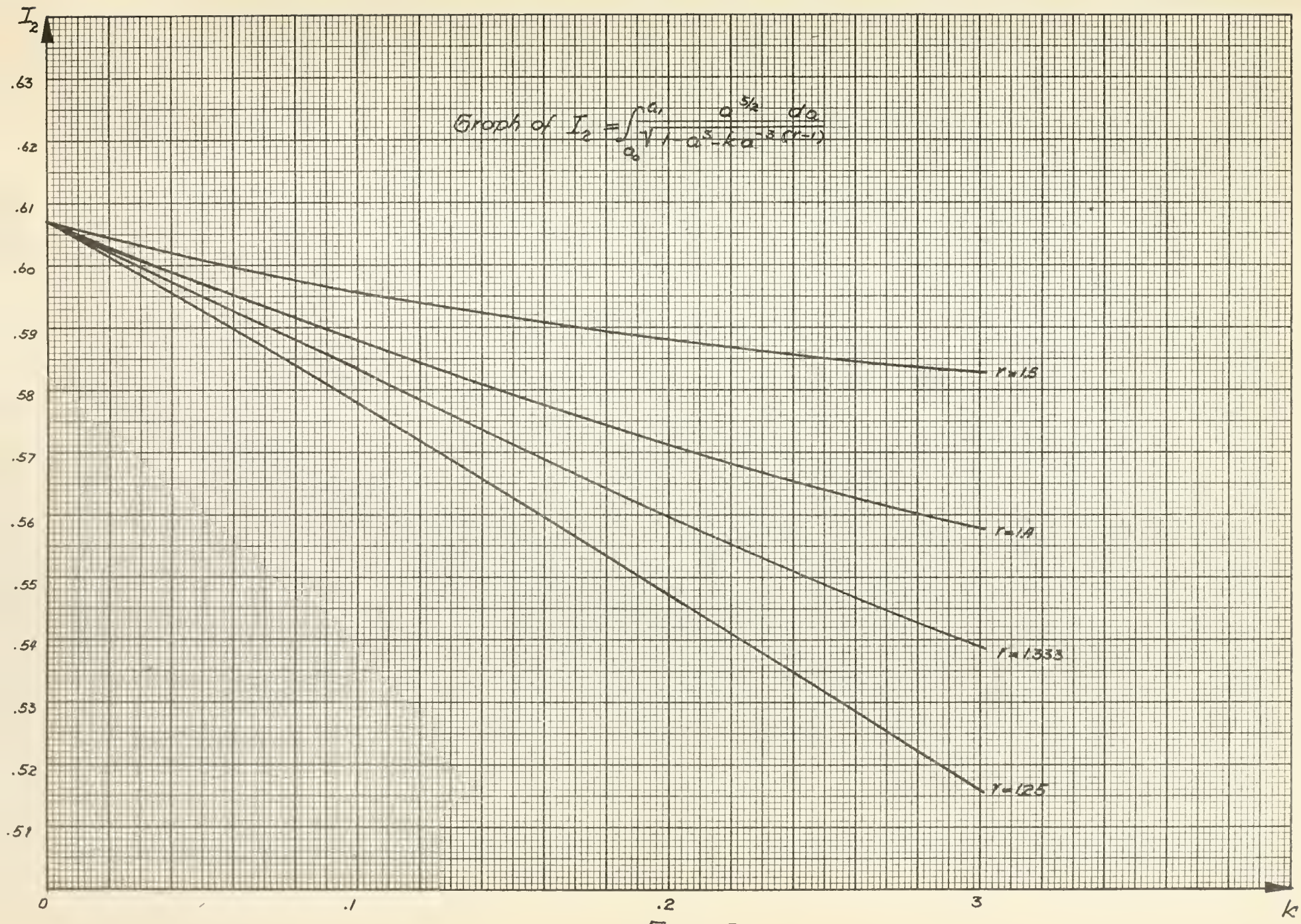

I_2
.63
.62
.61
.60
.59
.58
.57
.56
.55
.54
.53
.52
.51
Graph of $I_2 = \int_{a_0}^{a_1} \frac{a^{5/2}\, da}{\sqrt{1 - a^3 - ka^{-3(r-1)}}}$
r = 1.5
r = 1.4
r = 1.333
r = 1.25
0
.1
.2
3
k

Figure 2

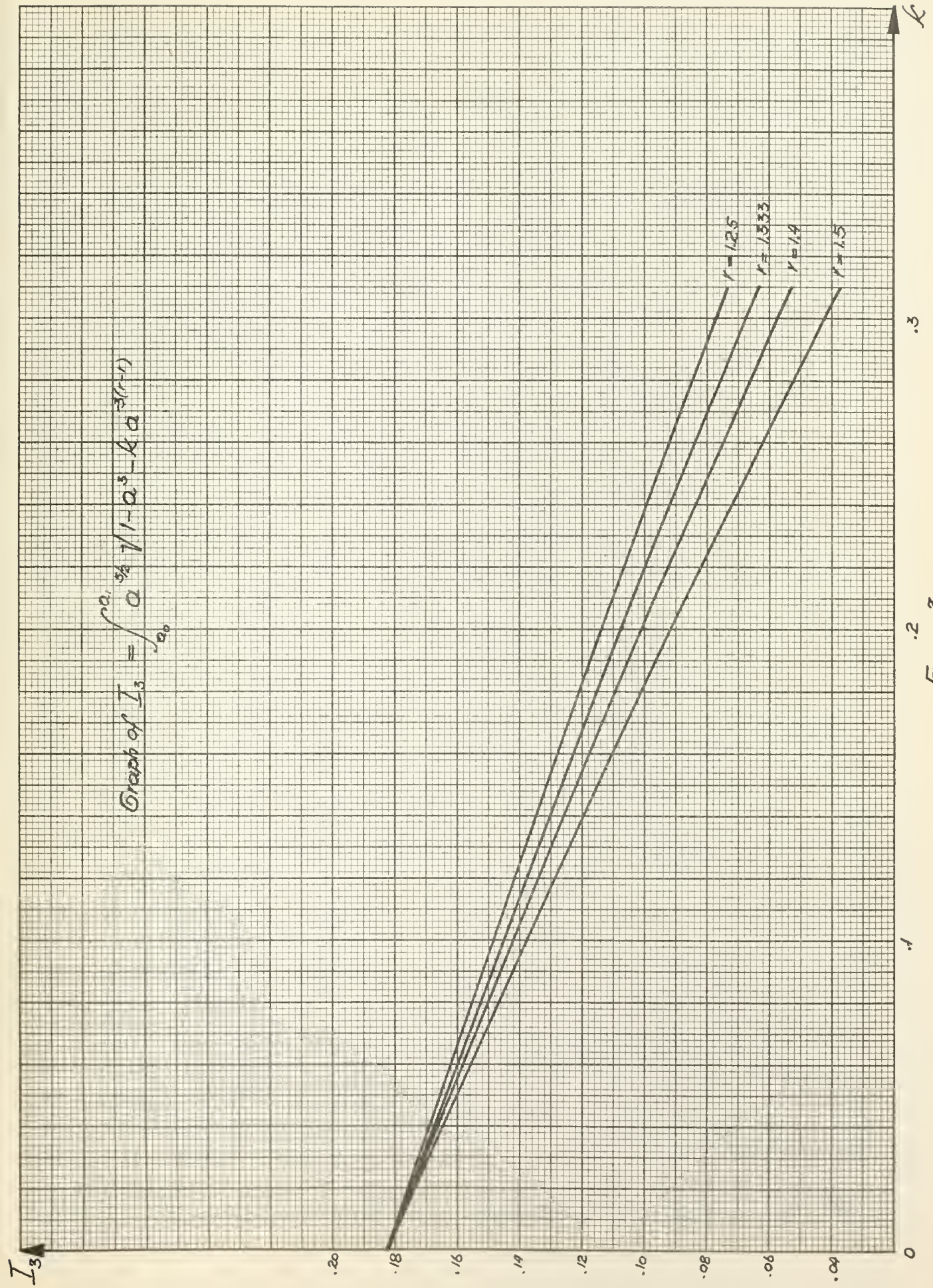

Graph of $I_3 = \int_{a_0}^{a_1} a^{\frac{3}{2}} \sqrt{1-a^3-ka^{3(r-1)}}$
r=1.25
r=1.333
r=1.4
r=1.5
I_3
k
.20
.18
.16
.14
.12
.10
.08
.06
.04
0
.1
.2
.3

Figure 3

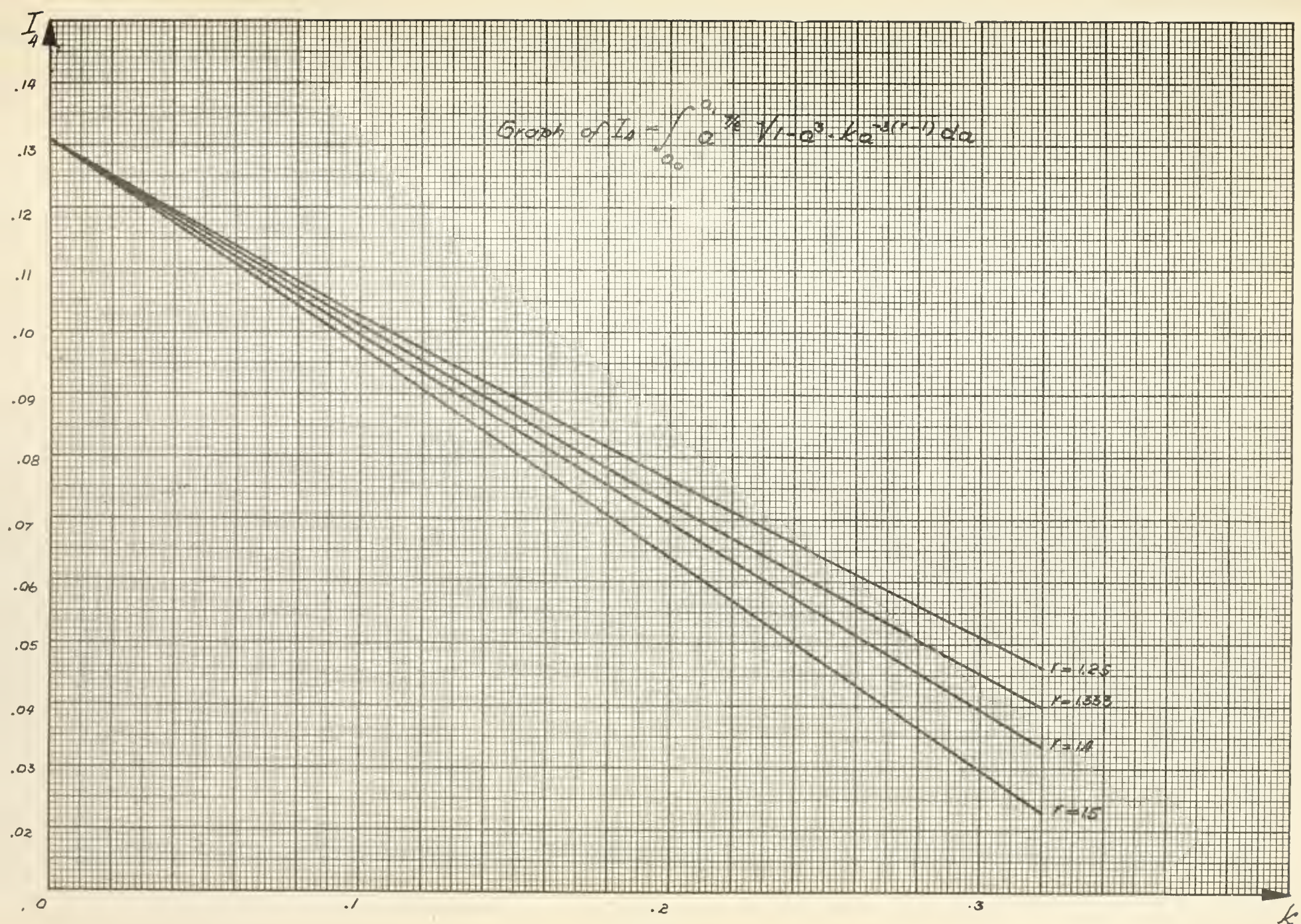

I_4
.14
.13
.12
.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.0
.1
.2
.3
k
Graph of $I_4 = \int_{a_0}^{a_1} a^{\frac{1}{2}} \sqrt{1-a^3 \cdot k^2}^{-\frac{1}{2}(r-1)} da$
r = 1.25
r = 1.333
r = 1.4
r = 1.5

Figure 4

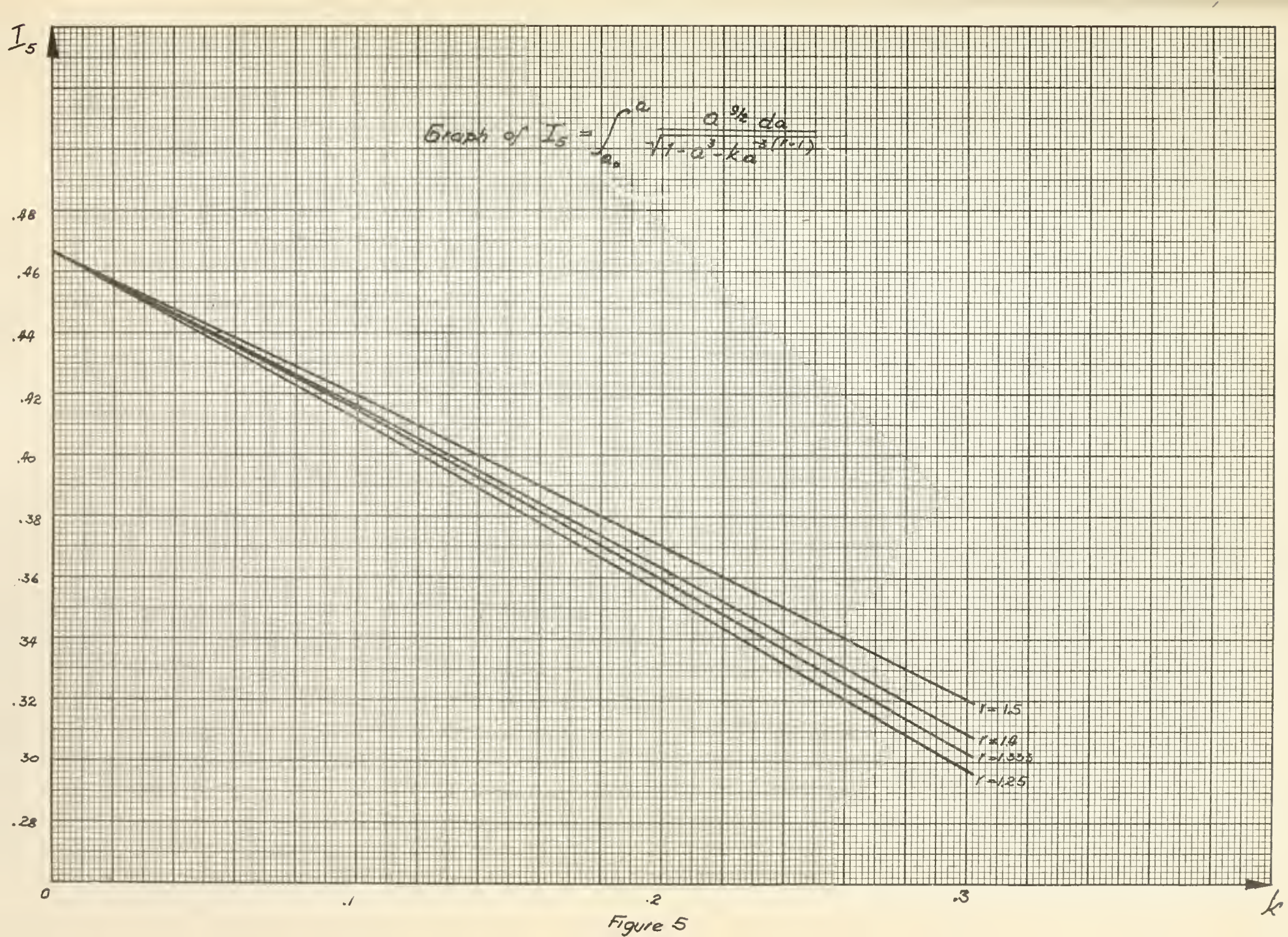

I_5
Graph of $I_5 = \int_{a_0}^{a} \frac{a^{9/2}\,da}{\sqrt{1-a^3-ka^{3(r-1)}}}$
.48
.46
.44
.42
.40
.38
.36
.34
.32
.30
.28
0
.1
.2
.3
k
r = 1.5
r = 1.4
r = 1.333
r = 1.25

Figure 5

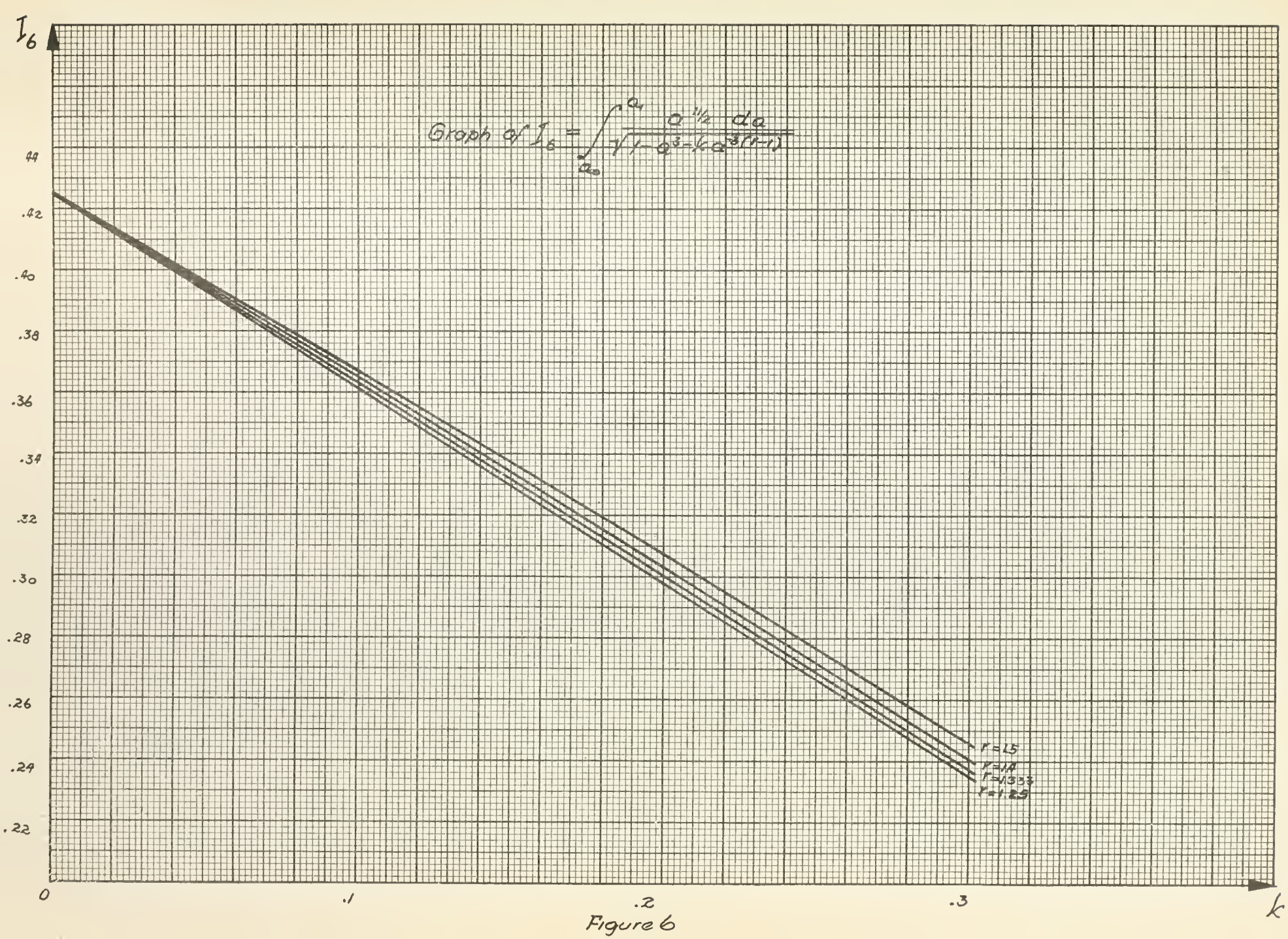

I_6
Graph of $I_6 = \int_{a_0}^{a_1} \frac{a^{1/2}\,da}{\sqrt{1-a^3-k\,a^{3(r-1)}}}$
.44
.42
.40
.38
.36
.34
.32
.30
.28
.26
.24
.22
0
.1
.2
.3
k
r=1.5
r=1.4
r=1.333
r=1.25

Figure 6

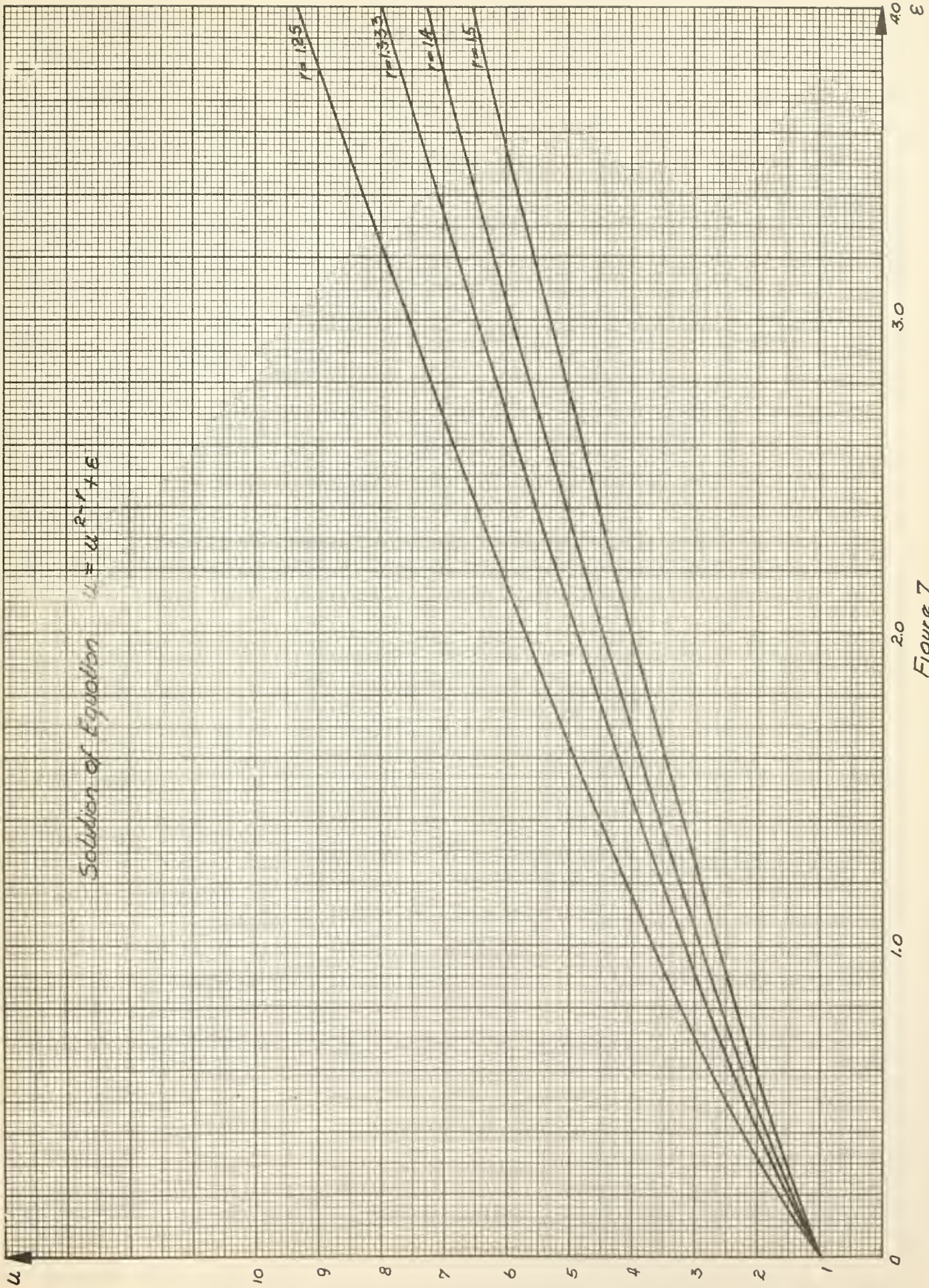

Solution of Equation u = u^2 - r + ε
r = 1.25
r = 1.375
r = 1.4
r = 1.5
u
ε
0
1.0
2.0
3.0
4.0
1
2
3
4
5
6
7
8
9
10

Figure 7

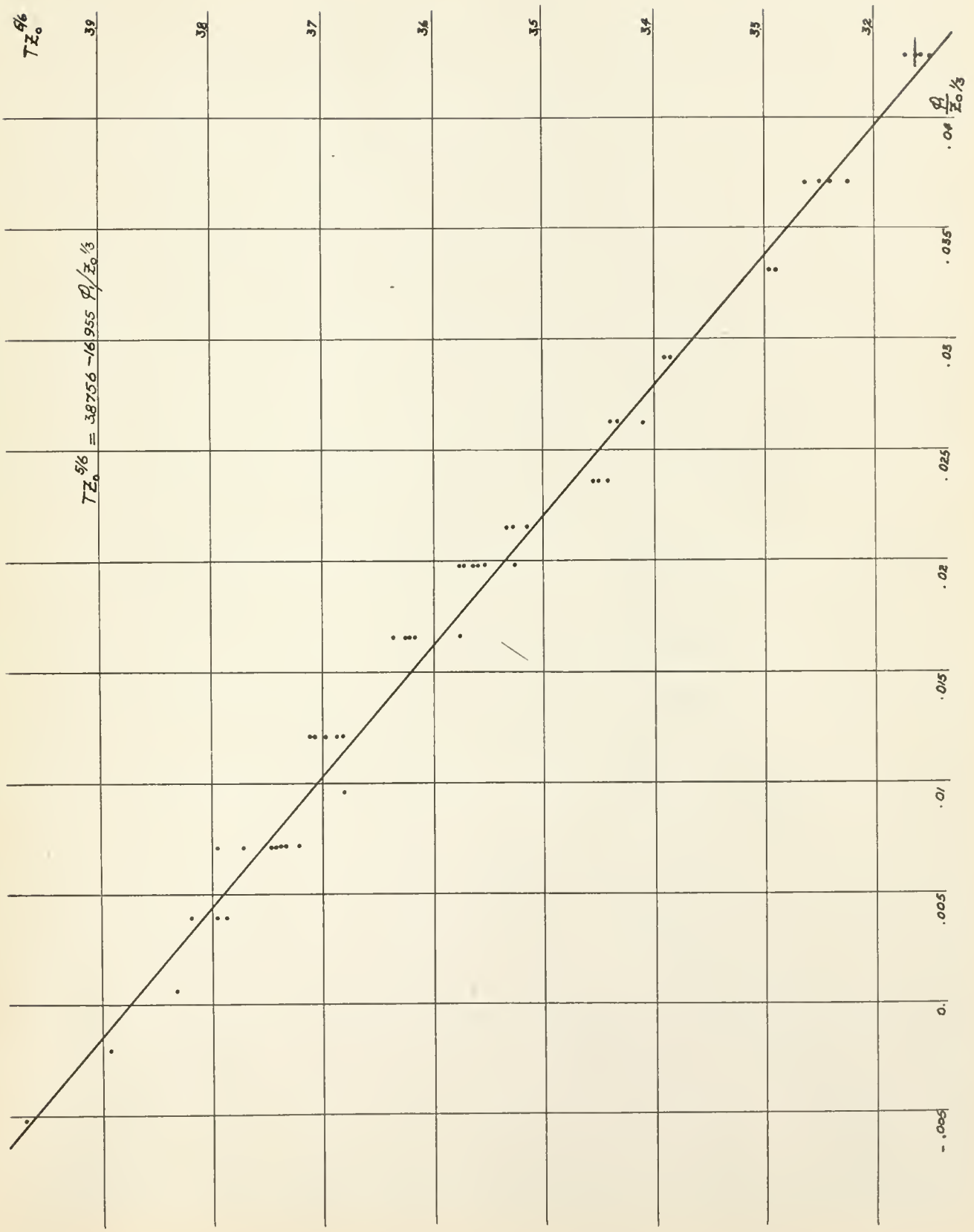

$TZ_o^{6/6}$
39
38
37
36
35
34
33
32
$TZ_o^{6/6} = 38756 - 16955\ P/Z_o^{1/3}$
-.005
0.
.005
.01
.015
.02
.025
.03
.035
.04
$P/Z_o^{1/3}$

Figure 8

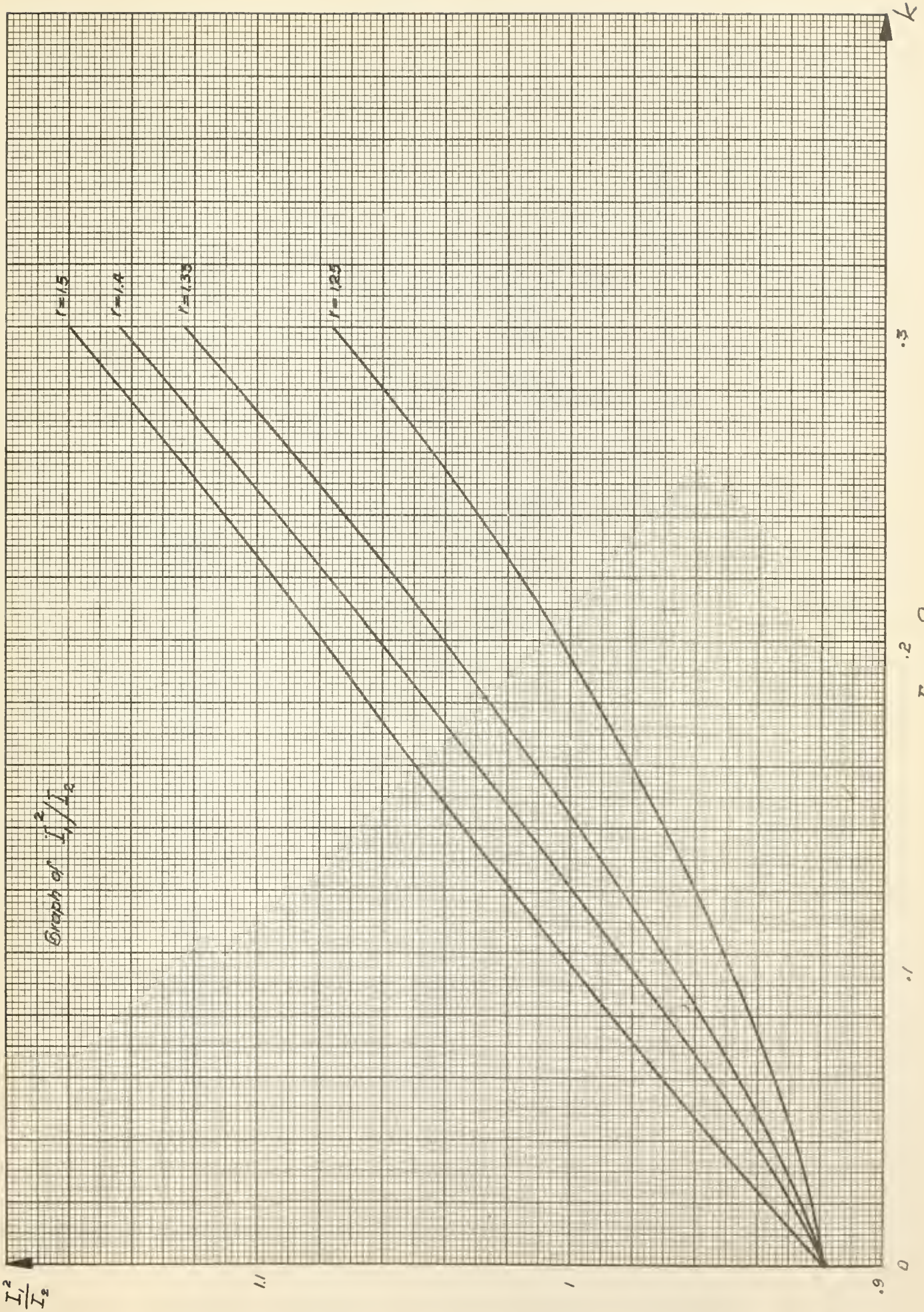


Figure 9

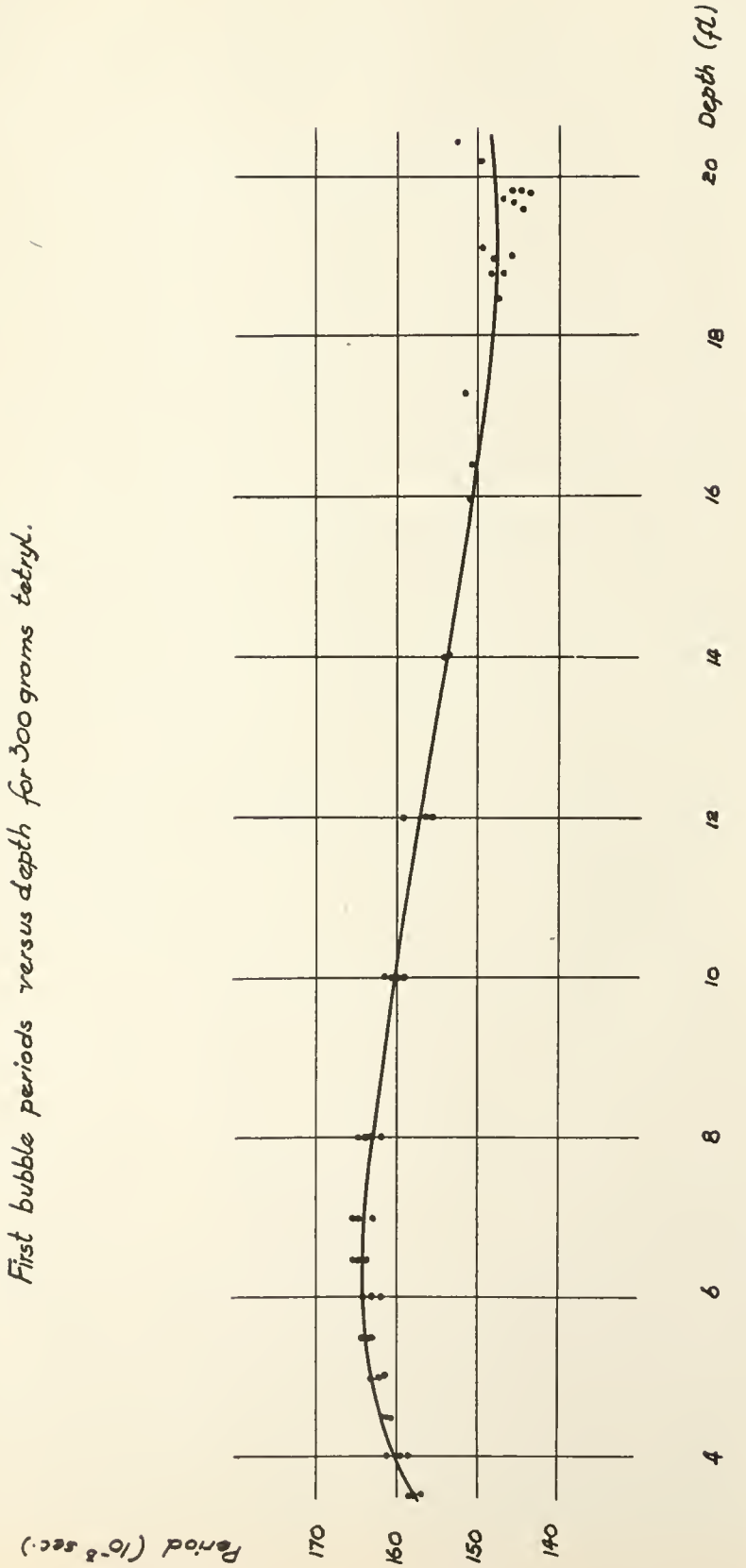

First bubble periods versus depth for 300 grams tetryl.
Period (10^{-8} sec.)
170
160
150
140
4
6
8
10
12
14
16
18
20
Depth (ft.)

Figure 10

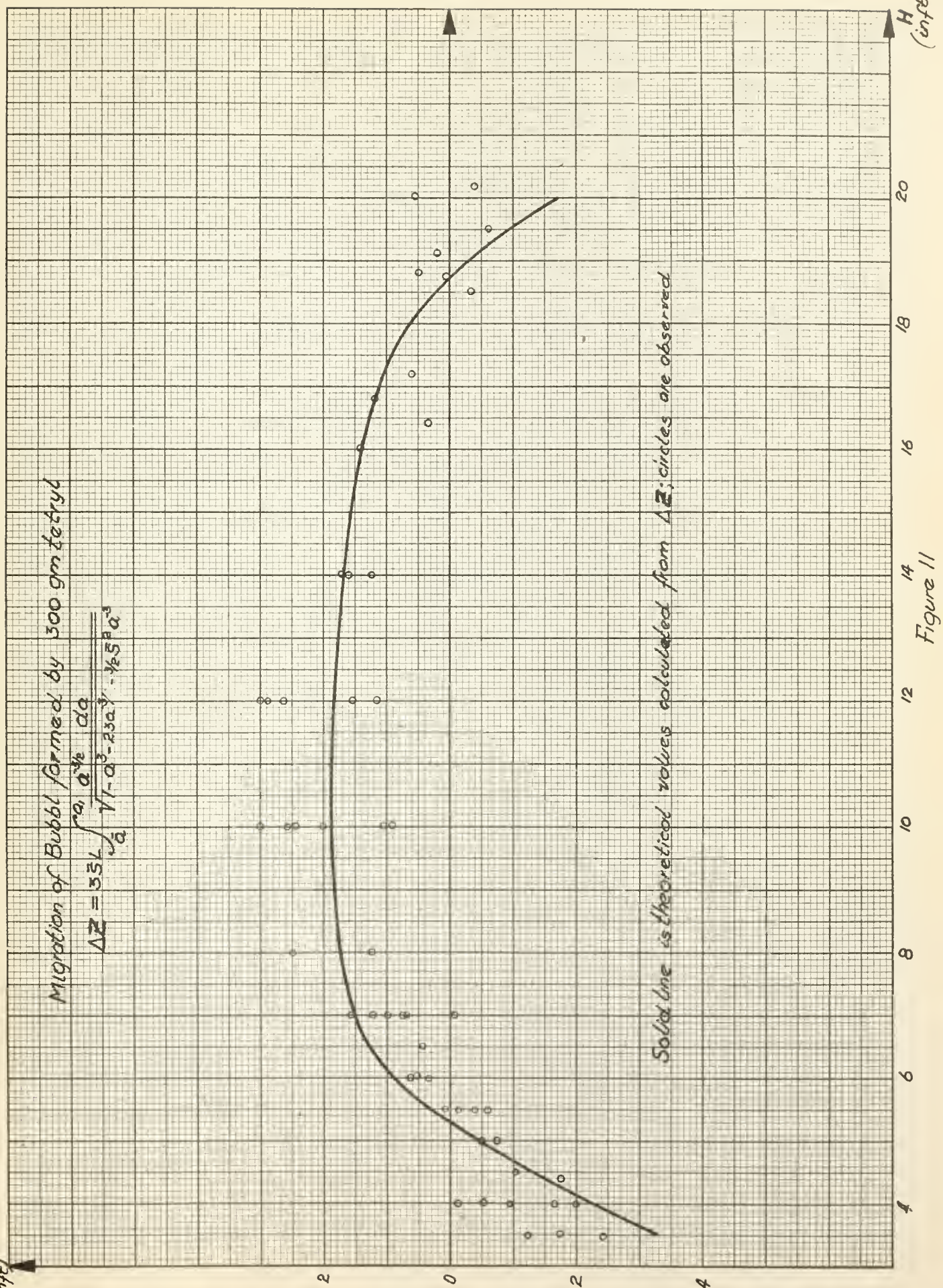

Migration of Bubbl formed by 300 gm tetryl
$\Delta Z = 33L \int_{a_1}^{a_2} \frac{a^{5/2}\, da}{\sqrt{1-a^3-23a^{\frac{1}{2}}-\frac{1}{2}S^2a^{-3}}}$
Solid line is theoretical values calculated from ΔZ; circles are observed
ΔZ(in ft)
H (in ft)
2
0
-2
-4
4
6
8
10
12
14
16
18
20

Figure 11

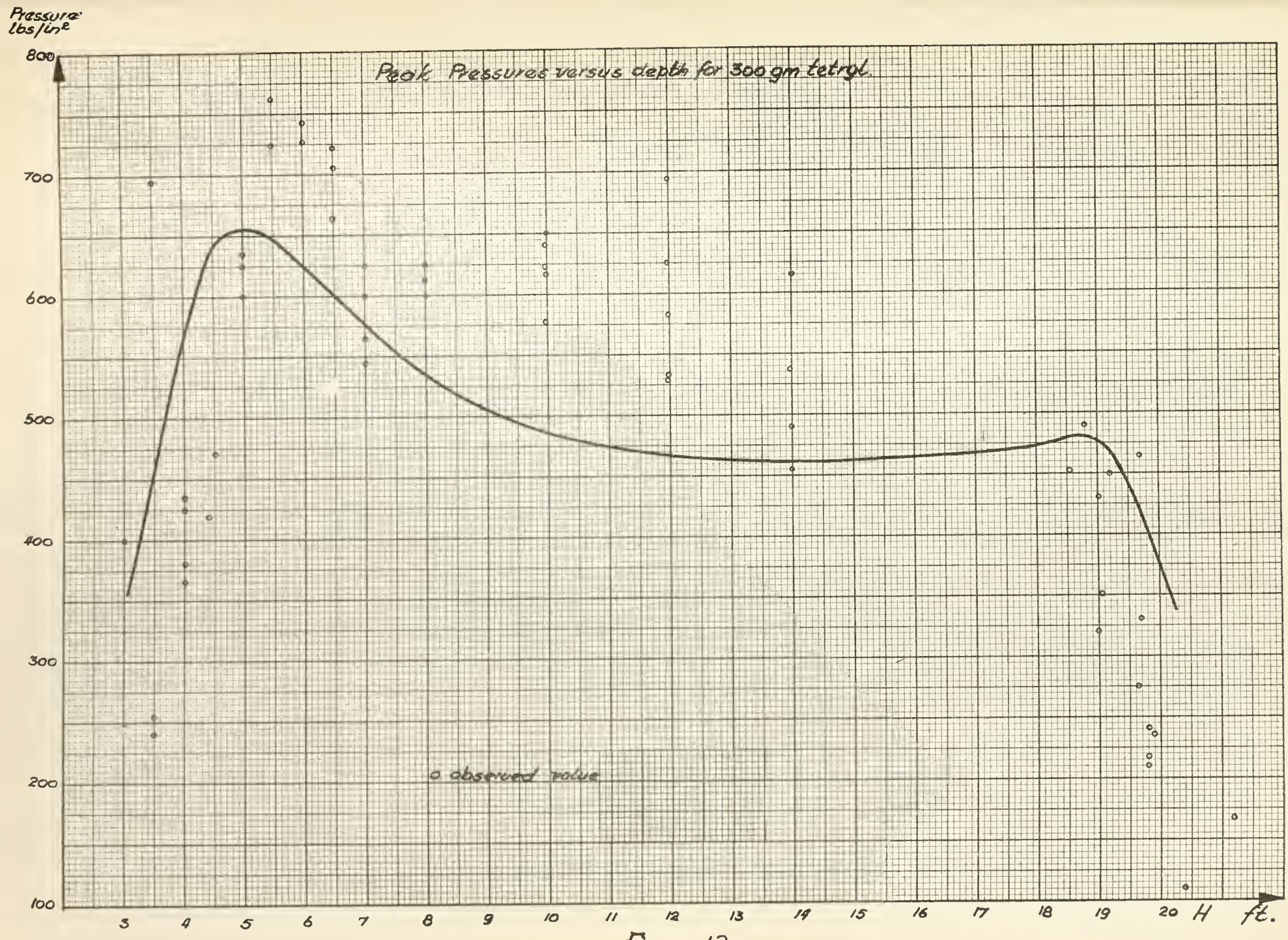
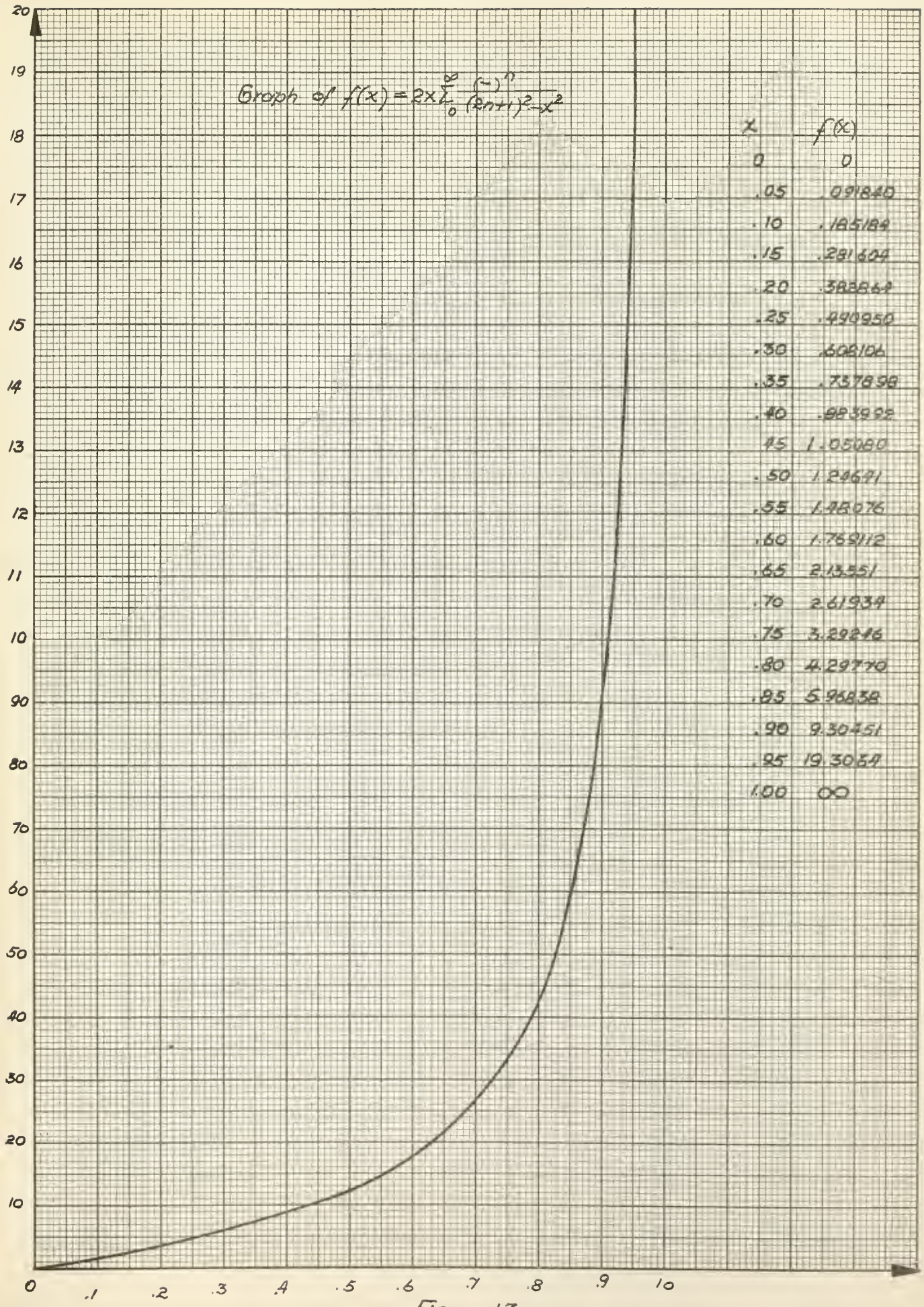

Pressure lbs/in²
Peak Pressures versus depth for 300 gm tetryl.
800
700
600
500
400
300
200
100
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
H ft.
o observed value

Figure 12



x	$f(x)$
0	0
.05	.091840
.10	.185184
.15	.281604
.20	.388862
.25	.490950
.30	.608106
.35	.737898
.40	.883992
.45	1.050880
.50	1.24671
.55	1.48076
.60	1.769112
.65	2.13551
.70	2.61934
.75	3.29246
.80	4.29770
.85	5.96838
.90	9.30451
.95	19.3054
1.00	∞

Figure 13

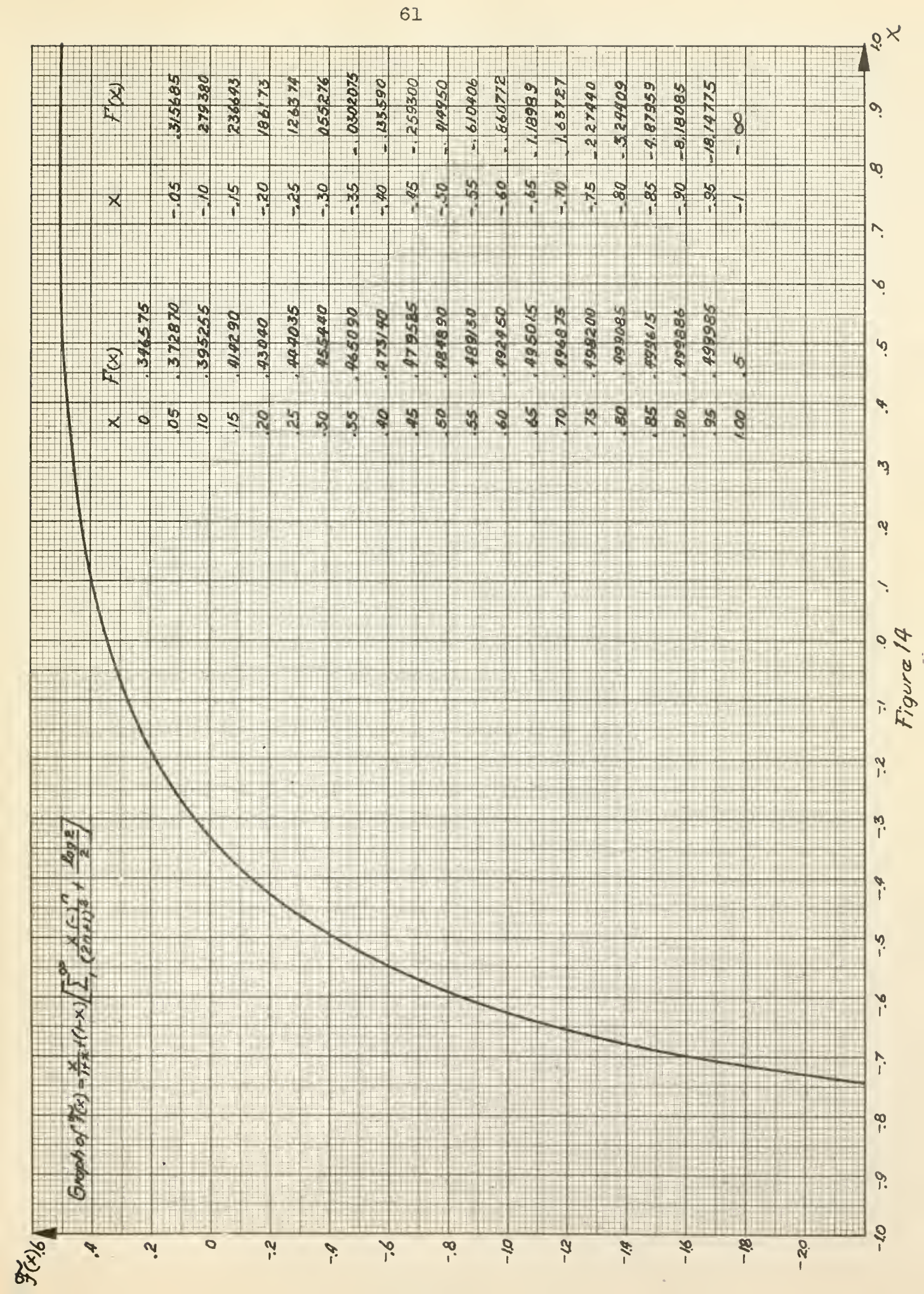
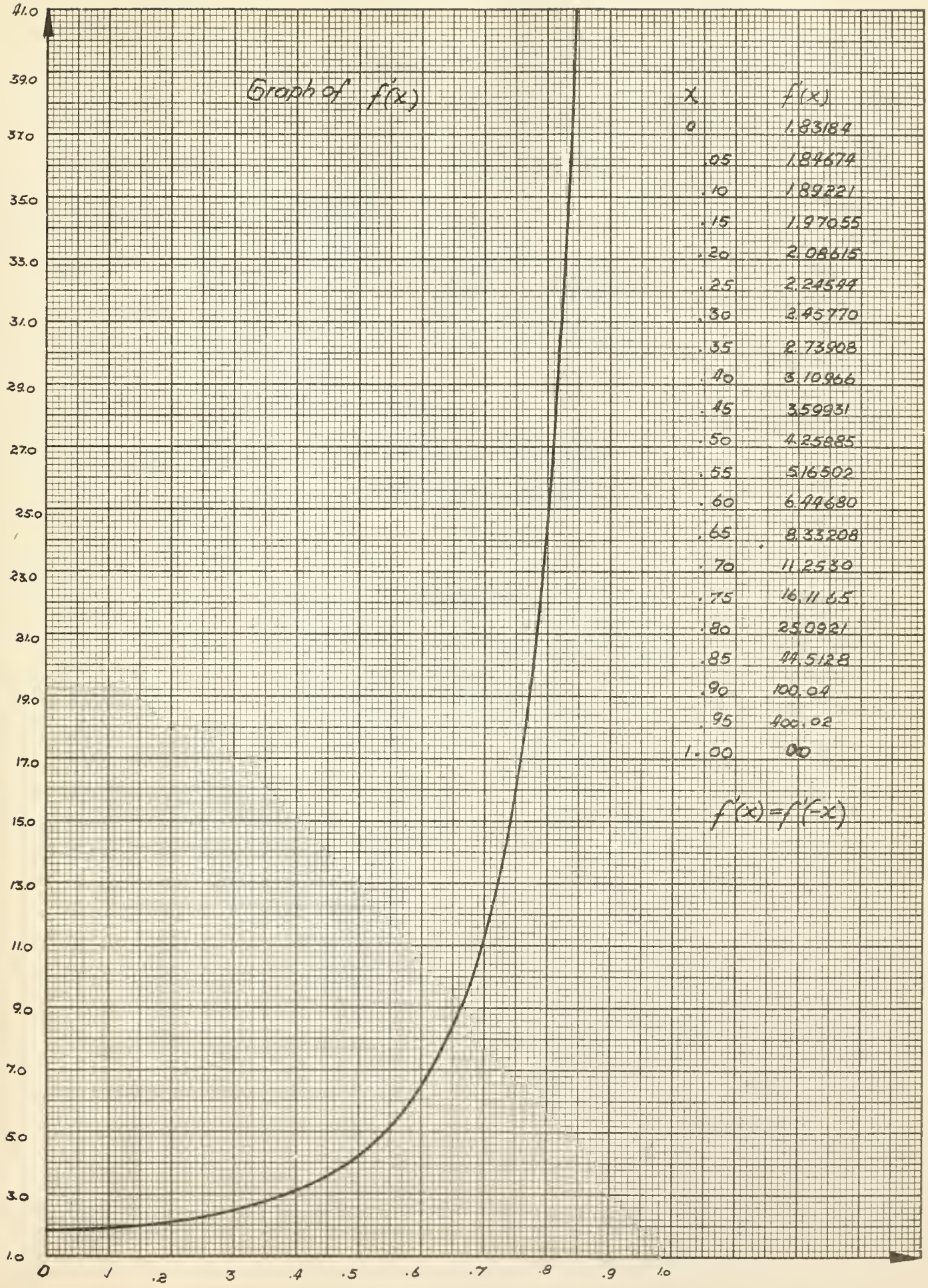

Graph of $F(x) = \frac{x}{\Gamma(\frac{1}{2})\Gamma(1-x)} \sum_{n=0}^{\infty} \frac{x(x-1)\cdots(x-n)}{(2n+1)3^n} + \frac{4\pi^2}{2}$
F(x)
X
F(x)
0 .346575
.05 .372870
.10 .395255
.15 .414290
.20 .43040
.25 .447035
.30 .455440
.35 .465090
.40 .473140
.45 .479585
.50 .484890
.55 .489150
.60 .492460
.65 .495015
.70 .496875
.75 .498200
.80 .499085
.85 .499615
.90 .499886
.95 .499985
1.00 .5
X
-.05 .315685
-.10 .279380
-.15 .236693
-.20 .186173
-.25 .126374
-.30 .055276
-.35 -.0302075
-.40 -.135590
-.45 -.259300
-.50 -.409950
-.55 -.610406
-.60 -.660772
-.65 -1.18989
-.70 -1.68727
-.75 -2.27440
-.80 -3.24409
-.85 -4.87959
-.90 -8.18085
-.95 -18.14775
-1 -∞
F(x)
X

Figura 14



x	f'(x)
0	1.83184
.05	1.84674
.10	1.89221
.15	1.97055
.20	2.08615
.25	2.24597
.30	2.45770
.35	2.73908
.40	3.10966
.45	3.59931
.50	4.25885
.55	5.16502
.60	6.44680
.65	8.33208
.70	11.2530
.75	16.1165
.80	25.0921
.85	44.5128
.90	100.04
.95	400.02
1.00	∞

$f'(x) = f'(-x)$

Figure 15

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